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About some commutative operations with classes of subcategories

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Abstract. Two operations on the class of subcategories of some categories are defined and conditions are indicated when these operations are commutative. Examples of factorization structures are given, whose subcategory classes satisfy the necessary conditions for these operations applied to them to be commutative.
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Keywords: the reflective, coreflective subcategories, the pair of conjugate subcategories, the factorization structures, category of locally convex topological vector spaces.

Despre unele operații comutative cu clase de subcategorii

Rezumat. Sunt definite două operații asupra claselor de subcategorii ale unor categorii și sunt indicate condițiile în care aceste operații sunt comutative. Sunt date exemple de structuri de factorizare, ale căror clase de subcategorii îndeplinesc condițiile necesare pentru ca aceste operații aplicate asupra lor să fie comutative.
Cuvinte-cheie: subcategorii reflective, coreflective, perechi de subcategorii conjugate, structuri de factorizare, categoria spațiilor local convexe topologice vectoriale.

1. Introduction

In this section introduces the notation and basic concepts that will be used throughout the paper. We recall standard definitions of subobjects and factor objects determined by classes of morphisms, as well as the associated classes of reflective and coreflective subcategories. Furthermore, we introduce pairs of conjugate subcategories and several notions of stability with respect to subobjects and factor objects, together with their corresponding dual notions.

Notations: Let C be a subcategory and $\mathcal{I}$ a class of monomorphisms. Then $\mathcal{S}_{\mathcal{I}}(C)$ is the subcategory of all $\mathcal{I}$-subobjects of objects in C. Dual: $\mathcal{Q}_{\mathcal{E}}(C)$.

Let $\mathbb{R}$ be the class of non-zero reflective subcategories, $\mathbb{K}$ - the class of non-zero coreflective subcategories of category $C_2\mathcal{V}$ of locally convex topological vector spaces, $\mathcal{R} \in \mathbb{R}$, $\mathcal{K} \in \mathbb{K}$ and the corresponding functors $r : C_2\mathcal{V} \to \mathcal{R}$ and $k : C_2\mathcal{V} \to \mathcal{K}$.

Let be $\varepsilon\mathcal{R} = \{e \in \mathcal{E}pi | r(e) \in \mathcal{I}so\}$, $\mu\mathcal{K} = \{m \in \mathcal{M}ono | k(m) \in \mathcal{I}so\}$. Both $\mu\mathcal{K}$ and $\varepsilon\mathcal{R}$ are classes of bimorphisms (see [1,2]).

$e : X \to Y \in \varepsilon\mathcal{R} \Leftrightarrow r^X = f \cdot e$ for a morphism f;

$m : X \to Y \in \mu\mathcal{K} \Leftrightarrow k^Y = m \cdot f$ for a morphism f.

Definition 1.1. *([1]) Let $\mathcal{R} \in \mathbb{R}$ and $\mathcal{K} \in \mathbb{K}$ be. The pair $(\mathcal{K}, \mathcal{R})$ is called a pair of conjugate subcategories if $\varepsilon\mathcal{R} = \mu\mathcal{K}$. These classes $\varepsilon\mathcal{R}$ (or $\mu\mathcal{K}$) form the class Bic of bicomplete classes.*

Let $\mathcal{E} \in \mathcal{E}pi$. Then $\mathbb{V}(\mathcal{E})$ is the class of reflective subcategories closed with respect to $\mathcal{E}$-factorobjects. Dual notion: $\mathbb{V}^c(\mathcal{M})$.

$\mathbb{V}^s(\mathcal{B})$ is the class of reflective subcategories of $\mathbb{V}(\mathcal{B})$ closed with respect to $\mathcal{B}$-subobjects. Dual notion: $\mathbb{V}_f(\mathcal{B})$. $\mathbb{V}^s_f(\mathcal{B}) = \mathbb{V}^s(\mathcal{B}) \cap \mathbb{V}_f(\mathcal{B})$.

$\mathbb{R}^s(\mathcal{B})$ is the class of reflective subcategories closed with respect to $\mathcal{B}$-subobjects. Dual notion: $\mathbb{R}_f(\mathcal{B})$. $\mathbb{R}^s_f(\mathcal{B}) = \mathbb{R}^s(\mathcal{B}) \cap \mathbb{R}_f(\mathcal{B})$.

$\mathbb{V}(\mathcal{B}, \mathcal{E}) = \mathbb{R}(\mathcal{B}) \bigcap \mathbb{V}(\mathcal{E})$.

$\mathbb{V}^s(\mathcal{B}, \mathcal{E}) = \mathbb{R}^s(\mathcal{B}) \bigcap \mathbb{V}(\mathcal{E})$.

$\mathbb{V}_f(\mathcal{B}, \mathcal{E}) = \mathbb{R}_f(\mathcal{B}) \bigcap \mathbb{V}(\mathcal{E})$.

$\mathbb{V}^s_f(\mathcal{B}, \mathcal{E}) = \mathbb{V}^s(\mathcal{B}, \mathcal{E}) \bigcap \mathbb{V}_f(\mathcal{B}, \mathcal{E})$.

Dual notions: $\mathbb{K}(\mathcal{B})$, $\mathbb{K}^s(\mathcal{B})$, $\mathbb{K}_f(\mathcal{B})$, $\mathbb{K}^s_f(\mathcal{B})$, $\overline{\mathbb{V}}^c(\mathcal{M})$, $\overline{\mathbb{V}}^s(\mathcal{B}, \mathcal{M})$, $\overline{\mathbb{V}}_f(\mathcal{B}, \mathcal{M})$, $\overline{\mathbb{V}}^s_f(\mathcal{B}, \mathcal{M})$.

2. Commutative operations with subcategory classes

In this section we investigate the commutativity of the operators $(\mathcal{S}_\mathcal{I})$ and $(\mathcal{Q}_\mathcal{E})$ acting on classes of subcategories. Under appropriate stability assumptions on the injection class $(\mathcal{I})$ and the projection class $(\mathcal{E})$ arising from left and right factorization structures, we show that the formation of subobjects and factor objects commutes, that is, ($\mathcal{Q}_\mathcal{E}\mathcal{S}_\mathcal{I}(C) = \mathcal{S}_\mathcal{I}\mathcal{Q}_\mathcal{E}(C)$). Several consequences for reflective and coreflective subcategories are then derived.

Theorem 2.1. *Let $(\mathcal{P}, \mathcal{I})$ a left factorization structure with injection class $\mathcal{I}$ stable on the right, $(\mathcal{E}, \mathcal{M})$ a right factorization structure with projection class $\mathcal{E}$ stable on the left, and $C \subset C_2\mathcal{V}$. Then*

$$\mathcal{Q}_\mathcal{E}\mathcal{S}_\mathcal{I}(C) = \mathcal{S}_\mathcal{I}\mathcal{Q}_\mathcal{E}(C).$$

Proof. $\Rightarrow$ $\mathcal{Q}_\mathcal{E}\mathcal{S}_\mathcal{I}(C) \subset \mathcal{S}_\mathcal{I}\mathcal{Q}_\mathcal{E}(C)$. Let $Z \in |\mathcal{Q}_\mathcal{E}\mathcal{S}_\mathcal{I}(C)|$ be. There is then an object $A \in \|C\|$ and the morfisms $i : X \to A \in \mathcal{I}$, $e : X \to Z \in \mathcal{E}$.

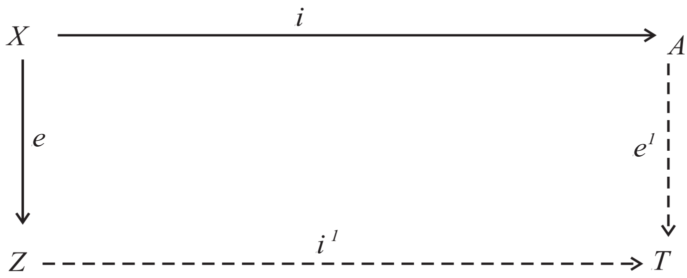

Figure 1. The puschout square built on the morphisms i and e.

Let be

$$i' \cdot e = e' \cdot i \tag{1}$$

the puschout square built on the morphisms i and e. Then $e' \in \mathcal{E}$ and $i' \in \mathcal{I}$ and $Z \in |\mathcal{S}_{\mathcal{I}}\mathcal{Q}_{\mathcal{E}}(\mathcal{C})|$.

$\Leftarrow$ $\mathcal{S}_{\mathcal{I}}\mathcal{Q}_{\mathcal{E}}(\mathcal{C}) \subset \mathcal{Q}_{\mathcal{E}}\mathcal{S}_{\mathcal{I}}(\mathcal{C})$. Dual demonstration. □

Corollary 2.1. *1. Let $(\mathcal{P}, \mathcal{I})$ a left factorization structure with stable injection class on the right, $(\mathcal{E}, \mathcal{M})$ a right factorization structure with stable projection class on the left, $\mathcal{R} \in \mathbb{R}$ and $\mathcal{K} \in \mathbb{K}$. Then*

$$a)\ \mathcal{Q}_{\mathcal{E}}\mathcal{S}_{\mathcal{I}}(\mathcal{R}) = \mathcal{S}_{\mathcal{I}}\mathcal{Q}_{\mathcal{E}}(\mathcal{R}),\quad b)\ \mathcal{Q}_{\mathcal{E}}\mathcal{S}_{\mathcal{I}}(\mathcal{K}) = \mathcal{S}_{\mathcal{I}}\mathcal{Q}_{\mathcal{E}}(\mathcal{K}).$$

2. Let $(\mathcal{E}_u, \mathcal{M}_p)$ = (the class of universal epimorphisms, the class of exact monomorphisms) = (the class of surjective morphisms, the class of topological inclusions);

$(\mathcal{E}_p, \mathcal{M}_u)$ = (the class of exact epimorphisms, the class of universal monomorphisms) (see [3,4]). Then:

$$a)\ \mathcal{Q}_{\mathcal{E}_u}\mathcal{I}_{\mathcal{M}_u}(\mathcal{R}) = \mathcal{I}_{\mathcal{M}_u}\mathcal{Q}_{\mathcal{E}_u}(\mathcal{R}),\quad b)\ \mathcal{Q}_{\mathcal{E}_u}\mathcal{I}_{\mathcal{M}_p}(\mathcal{R}) = \mathcal{I}_{\mathcal{M}_p}\mathcal{Q}_{\mathcal{E}_u}(\mathcal{R}).$$

3. Some properties of operations with subcategory classes

In this section we analyze structural properties of the operations defined on classes of subcategories. In particular, we examine the behavior of the operators $(\mathcal{Q}_{\mathcal{B}})$ and $(\mathcal{S}_{\mathcal{B}})$, if $\mathcal{B}$ is a bicomplete class of morphisms, with respect to products, subobjects, and factorizations associated with classes of morphisms. Theorem 3.1 establishes the commutativity of certain canonical diagrams associated with the operators $(\mathcal{Q}_{\mathcal{B}})$ and $(\mathcal{S}_{\mathcal{B}})$, as well as stability conditions with respect to products and subobjects. As applications, the subsequent corollaries highlight consequences for reflective and coreflective subcategories associated with left and right factorization structures.

Theorem 3.1. *Let $\mathcal{B} \in \mathbb{B}ic$ be. Then the following diagrams are commutative:*

a)

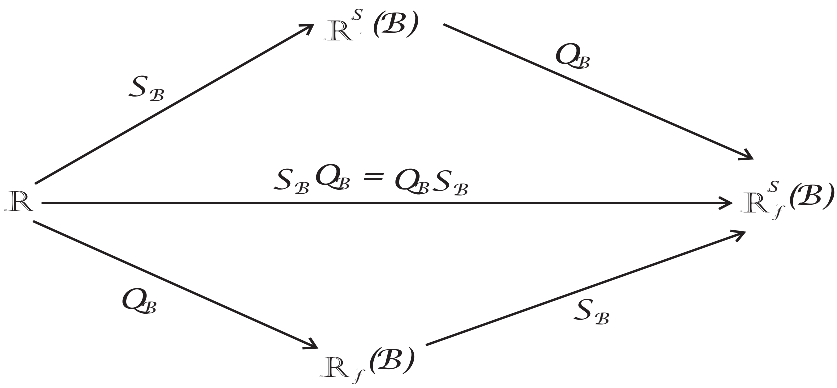

Figure 2. Commutativity of operations ($Q_{\mathcal{B}}$) and ($S_{\mathcal{B}}$) on class $\mathbb{R}$.

b)

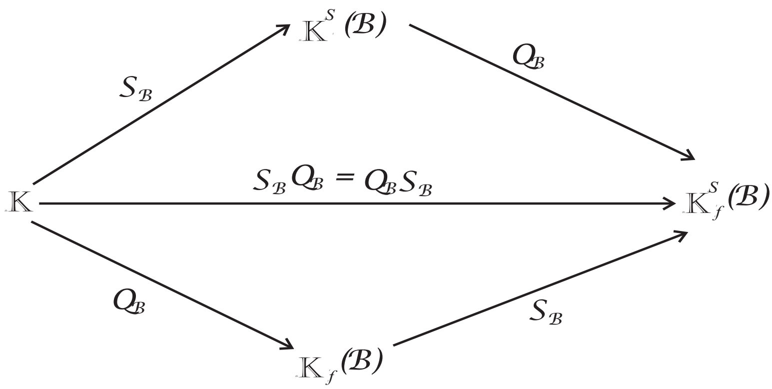

Figure 3. Commutativity of operations ($Q_{\mathcal{B}}$) and ($S_{\mathcal{B}}$) on class $\mathbb{K}$.

Proof. a) $Q_{\mathcal{B}}(\mathcal{R})$ is closed in relation to the products. Indeed, let $\{A_i | i \in \mathcal{I}\}$ be a family of objects in $Q_{\mathcal{B}}(\mathcal{R})$. Then there exist the objects $T_i \in |\mathcal{R}|$ and the morphisms $b_i : T_i \to A_i \in \mathcal{B}, i \in \mathcal{I}$. The morphism

$$b = \prod b_i : \prod T_i \to \prod A_i$$

belongs to class $\mathcal{B}$ and $\prod T_i \in |\mathcal{R}|$. So $\prod A_i \in |Q_{\mathcal{B}}(\mathcal{R})|$.

Let's verify that $Q_{\mathcal{B}}(\mathcal{R})$ is closed with respect to $(\varepsilon\mathcal{R})^{\perp}$-subobjects. Let $A \in |Q_{\mathcal{B}}(\mathcal{R})|$ and $m : X \to (\varepsilon\mathcal{R})^{\perp}$ be. There are $T \in |\mathcal{R}|$ and the morphism $b : T \to A \in \mathcal{B}$. We examine the pullback square:

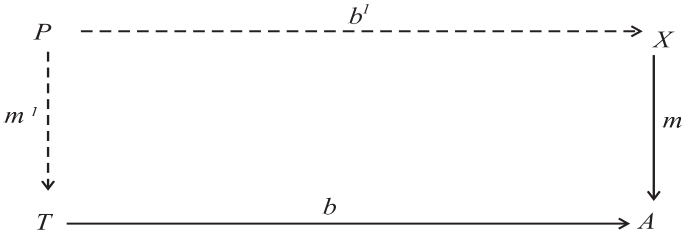

Figure 4. The pullback square built on the morphisms b and m

$$b \cdot m' = m \cdot b' \tag{1}$$

Then $b' \in \mathcal{B}$ and $m' \in (\varepsilon\mathcal{R})^{\perp}$. So $P \in \mathcal{R}$, and $X \in |Q_{\mathcal{B}}(\mathcal{R})|$. Thus we have proven that $Q_{\mathcal{B}}(\mathcal{R})$ is a $(\varepsilon\mathcal{R})$-reflective subcategory.

Let's verify that $\mathcal{S}_{\mathcal{B}}(\mathcal{R})$ is a $(\varepsilon\mathcal{R})$-reflective subcategory. Let be $X \in |C_2\mathcal{V}|$, $r^X : X \to rX$, his $\mathcal{R}$ - replica,

$$r^X = b^X \cdot r_1^X \tag{2}$$

$(\mathcal{B}^{\top}, \mathcal{B})$-factorization of morphism r^X and we will demonstrate that $r_1^X : X \to r_1 X$ is $\mathcal{S}_{\mathcal{B}}(\mathcal{R})$-replica of object X. Let $Z \in |\mathcal{S}_{\mathcal{B}}(\mathcal{R})|$ and $f : X \to Z$. It exists $T \in |\mathcal{R}|$ and $b : Z \to T \in \mathcal{B}$. Then

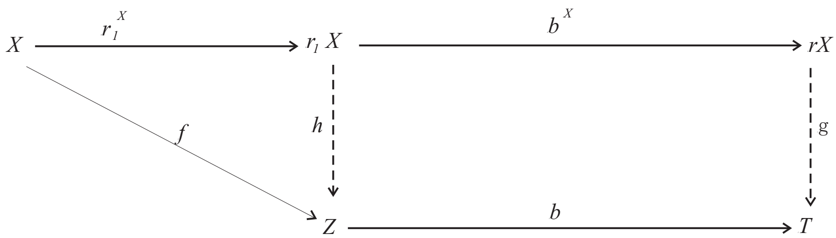

Figure 5. r_1^X is $\mathcal{S}_{\mathcal{B}}(\mathcal{R})$-replica of object X.

$$b \cdot f = g \cdot r^X \tag{3}$$

or

$$b \cdot f = g \cdot b^X \cdot r_1^X \tag{4}$$

for un morphism g. Because $r_1^X \perp b$, we deduce that

$$f = h \cdot r_1^X \tag{5}$$

Thus f is extended by r_1^X.

b) Dual demonstration. □

Corollary 3.1. *Let be $\mathcal{B} \in \mathbb{B}ic$, $(\mathcal{E}, \mathcal{M})$ a right factorization structure with projection class $\mathcal{E}$ stable on the left and $\mathcal{R} \in \mathbb{R}$. Then the following diagram is commutative.*

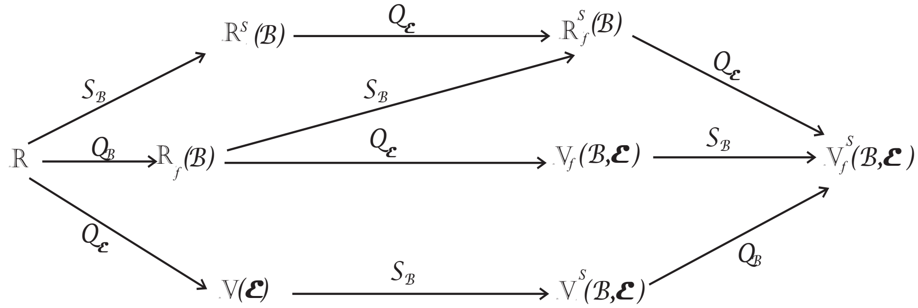

Figure 6. Commutativity of operations $(Q_\mathcal{B})$, $(\mathcal{S}_\mathcal{B})$ and $(\mathcal{S}_\mathcal{E})$ on class $\mathbb{R}$.

Corollary 3.2. *Let be $\mathcal{B} \in \mathbb{B}ic$, $(\mathcal{E}, \mathcal{M})$ a left factorization structure with bijection class $\mathcal{E}$ stable on the right and $\mathcal{K} \in \mathbb{K}$. Then the following diagram is commutative.*

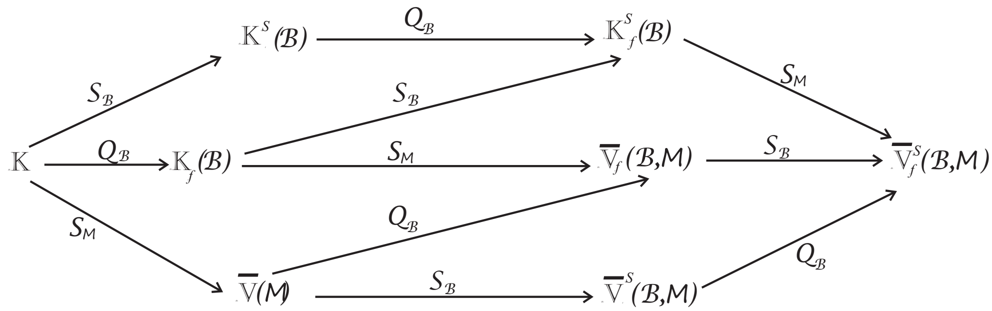

Figure 7. Commutativity of operations $(Q_\mathcal{B})$, $(\mathcal{S}_\mathcal{B})$ and $(\mathcal{S}_\mathcal{M})$ on class $\mathbb{K}$.

4. Conclusions

In this paper we have investigated the behavior of operations with classes of subcategories generated by subobjects and factor objects in the category $C_2\mathcal{V}$ of locally convex topological vector spaces. The main emphasis was placed on the interaction between the operators $\mathcal{S}_\mathcal{I}$ and $Q_\mathcal{E}$ associated with left and right factorization structures, as well as on their impact on reflective and coreflective subcategories.

First, we established sufficient stability conditions on the classes of monomorphisms and epimorphisms under which the formation of $\mathcal{I}$-subobjects and $\mathcal{E}$-factor objects is commutative. Theorem 2.1 shows that, in the presence of $\mathcal{I}$-right-stable injection classes and $\mathcal{E}$-left-stable projection classes, the equality

$$Q_{\mathcal{E}}S_{\mathcal{I}}(C) = \mathcal{S}_{\mathcal{I}}Q_{\mathcal{E}}(C),$$

holds for arbitrary subcategories C. This result provides a categorical framework in which subobject and factor object constructions can be interchanged without loss of generality. As a consequence, several commutativity properties were derived for reflective and coreflective subcategories, including important cases arising from classical factorization structures in $C_2\mathcal{V}$, such as those determined by universal morphisms ($\mathcal{M}_u$) and exact morphisms ($\mathcal{M}_p$). These results clarify the role of bicomplete classes and conjugate pairs of subcategories in the study of stability with respect to subobjects and factor objects.

In the second part of the paper, we analyzed structural properties of the operators $\mathcal{S}_{\mathcal{B}}$ and $Q_{\mathcal{B}}$ for bicomplete classes $\mathcal{B}$. Theorem 3.1 shows that these operators give rise to commutative canonical diagrams and preserve essential categorical constructions such as products and suitable classes of subobjects. In particular, we proved that $Q_{\mathcal{B}}(\mathcal{R})$ and $\mathcal{S}_{\mathcal{B}}(\mathcal{R})$ inherit reflectivity properties from $\mathcal{R}$, while their dual counterparts preserve coreflectivity. The obtained results unify and extend several known facts concerning reflective and coreflective subcategories in categories with factorization structures. They also provide a systematic approach to studying stability properties via bicomplete classes and conjugate subcategories. Further research may focus on applications of these constructions to more general topological or functional-analytic categories, as well as on the interaction with other categorical closure operators.

Open problems and perspectives. The results obtained in this paper naturally lead to the following open problems:

Problem 4.1 (Generality of commutativity). To what extent does the commutativity $Q_{\mathcal{E}}S_{\mathcal{I}}(C) = \mathcal{S}_{\mathcal{I}}Q_{\mathcal{E}}(C)$, remain valid in categories beyond $C_2\mathcal{V}$? In particular, can analogous results be established in quasi-abelian, exact, or locally presentable categories equipped with suitable factorization structures?

Problem 4.2 (Minimal stability assumptions). Are the stability conditions imposed on the injection and projection classes in Theorem 2.1 necessary? Determine minimal or alternative sets of assumptions on $\mathcal{I}$ and $\mathcal{E}$ ensuring the commutativity of the operators $\mathcal{S}_{\mathcal{I}}$ and $Q_{\mathcal{E}}$.

Problem 4.3 (Structure of bicomplete classes). Provide an intrinsic characterization and, if possible, a classification of bicomplete classes of morphisms in categories of topological or functional-analytic nature. How do such classes determine and reflect conjugate pairs of reflective and coreflective subcategories?

Problem 4.4 (Interaction with other categorical closures). Investigate the relationship between the operators $\mathcal{S}_{\mathcal{B}}$, $\mathcal{Q}_{\mathcal{B}}$ and other categorical closure operators, such as torsion theories, localizations, or reflective hulls. Under what conditions do these constructions coincide or interact in a nontrivial way?

Problem 4.5 (Functorial behavior and invariance). Study the behavior of the constructions introduced here under functorial changes of categories. In particular, determine which properties of bicomplete classes and conjugate subcategories are preserved under adjoint functors or equivalences of categories.

Problem 4.6 (Homological and topological applications). Explore potential connections between the presented framework and homological methods in topological vector spaces. For instance, can the stability results be used to characterize classes of spaces closed under exact sequences, limits, or colimits of a prescribed type?

Problem 4.7 (Concrete realizations in functional analysis). Identify concrete classes of locally convex spaces for which the abstract results of this paper yield new closure or stability properties with respect to subspaces, quotients, and products. In particular, investigate applications to classical categories such as barreled, bornological, or nuclear spaces.

These open problems indicate that the theory developed here opens several avenues for further research, both at the abstract categorical level and in concrete functional-analytic contexts.

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