Encoding the Information in a Quantum Computer Using Effective Spin Two-Boson Representation

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***Abstract* - The Schwinger representation of angular momentum is used to quantum computing. In this representation, examples of encoding and decoding of information in a quantum computer based on maximally entangled quantum states (Bell states) are presented.**

***Key words* - Qubit, quantum circuit, Hadamard gate, CNOT operator, encoding the information, Schwinger representation, entanglement, Bell states.**

I. INTRODUCTION

Quantum computing is based on application of linear operators, which transform the state vectors of one- and many-qubit systems in quantum circuits containing quantum logical elements. In this case, a ket-vector $|\psi\rangle$ of one qubit is defined in a two-dimensional Hilbert space with basis vectors $|0\rangle$ and $|1\rangle$ as a linear superposition of these vectors with arbitrary complex constants c_1 and c_2 ($|\psi\rangle = c_1|0\rangle + c_2|1\rangle$, $|c_1|^2 + |c_2|^2 = 1$). Two examples of one-qubit systems are an electron with two possible projections of the spin on the quantification axis and a photon with two possible polarizations.

The states of a *N*-qubit system are determined by the tensor product of corresponding one-qubit states: $|\psi\rangle = |\psi_1\rangle \otimes ... \otimes |\psi_N\rangle$, where $|\psi_1\rangle \in H_1, ..., |\psi_N\rangle \in H_N$, $|\psi\rangle \in H_1 \otimes ... \otimes H_N$ and H_K $(K = 1, ..., N)$ are the Hilbert spaces to which the state vectors $|\psi_K\rangle$ belong.

It would seem possible to store in a quantum computer an infinite amount of information, because any qubit is represented geometrically by the radius-vector of a point on the Bloch sphere surface. However, such a scenario cannot be realized because of the limitations imposed by the Wootters and Zurek no-cloning theorem [1, 2].

Despite the restrictions of the no-cloning theorem, there are a lot of publications on quantum computing, especially after the discovery of Deutsch-Jozsa [3, 4], Shor [5] and Grover [6, 7] algorithms. In these studies the spin algebra formalism is used due to its simplicity. Earlier, it was shown that the Schwinger representation of the angular momentum [8] can be used in quantum computing [9].

In this paper, encoding and decoding the information in a quantum computer in Schwinger representation are discussed.

II. TWO-BOSON REPRESENTATION OF AN EFFECTIVE SPIN RELATED TO A N-QUBIT SYSTEM

In the Schwinger representation, the spin projection operators of the spin $S = 1/2$ are [8]:

$$S_x = \frac{1}{2}(a^+b + ab^+), \quad S_y = \frac{i}{2}(ab^+ - a^+b),$$
$$S_z = \frac{1}{2}(a^+a - b^+b), \qquad (1)$$

where a^+, a and b^+, b are the creation and destruction operators of the bosons of *a*- and *b*-types, which satisfy the kinematic conditions

$$a^+a + b^+b = I, \qquad (2)$$

and I is the unit operator, defined in the two-dimension Hilbert space. Based on this relation, only lowest states of coupled *a*- and *b*-oscillators satisfy the condition $n_a + n_b = 1$. Only these lowest states are necessary for representing the two basis spin functions. Therefore, the infinite number of oscillators states with $n_a + n_b > 1$ must be excluded.

In the case of the effective half-integer spin $S = 2^{N-1} - \frac{1}{2}$ related to a *N*-qubit system, there are $2S$ kinematic conditions that follow from the unitarity of the spinor operator:

$$U_S = \begin{pmatrix} [(2S)!]^{-1/2}a^{2S} \\ [(2S-1)!]^{-1/2}a^{2S-1}b \\ \vdots \\ [(S+M)!\,(S-M)!]^{-1/2}a^{S+M}b^{S-M} \\ \vdots \\ [(2S-1)!]^{-1/2}ab^{2S-1} \\ [(2S)!]^{-1/2}b^{2S} \end{pmatrix}. \qquad (3)$$

Indeed, averaging the operator equation

$$U_S^+ U_S = I \qquad (4)$$

with respect to two-boson wave functions $|S+M\rangle_a|S-M\rangle_b$ $(M = S, S-1, ..., 1-S, -S)$, one can obtain the following algebraic equation of $2S$ degree with respect to the variable $n = n_a + n_b$ [9]:

$$n^{2S} - C_{2S-1}n^{2S-1} + C_{2S-2}n^{2S-2} - \cdots + (-1)^{2S}C_2n^2 + (-1)^{2S+1}\left[2\left(S - \tfrac{1}{2}\right)\right]!\,n = (2S)! \qquad (5)$$

Here *I* is the unit operator, defined in the (2*S*+1)-dimension Hilbert space,

$$|S+M\rangle_a|S-M\rangle_b =$$

$$[(S+M)!\,(S-M)!]^{-\frac{1}{2}}(a^+)^{S+M}(b^+)^{S-M}|0\rangle, \qquad (6)$$

and $|0\rangle$ is the vacuum state $\left(|0\rangle = |0\rangle_a|0\rangle_b\right)$. In the case of one qubit (S =1/2), the Eq. (5) transforms into the kinematic condition $n = n_a + n_b = 1$, which is the Eq. (2) averaged on two-boson wave functions $|1\rangle_a|0\rangle_b$ and $|0\rangle_a|1\rangle_b$. At half-integer S >1/2 there are 2S solutions of the Eq. (5), of which one is a real solution, $n_1 = 2S$, and the other 2S-1 solutions are either pairwise imaginary in the case of S = 3/2, or pairwise complex conjugated in all other cases.

III. Bell states

Let us consider a quantum circuit containing the *CNOT* gate, in the controlled entrance of which the one-qubit Hadamart element H is switched on. On both entrances (controlled and target) of the quantum circuit the same basis vectors $|1\rangle_a|0\rangle_b$ and $|0\rangle_a|1\rangle_b$ are switched on. After passing the Hadamart gate, one of these vectors becomes (Fig. 1a)

$$H|1\rangle_a|0\rangle_b = \frac{1}{\sqrt{2}}[a^+(a+b) + b^+(a-b)]\left(|1\rangle_a|0\rangle_b\right) = \frac{1}{\sqrt{2}}\left(|1\rangle_a|0\rangle_b + |0\rangle_a|1\rangle_b\right), \qquad (7)$$

where

$$H = \frac{1}{\sqrt{2}}[a^+(a+b) + b^+(a-b)]. \qquad (8)$$

The operators a^+, a, b^+ and b from (8) satisfy the kinematic condition (2).

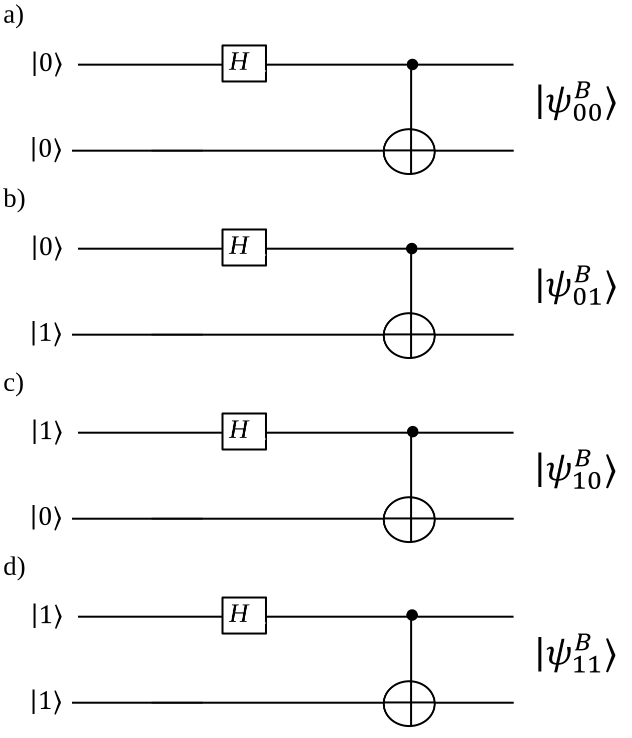

Fig. 1. Quantum circuit for encoding the information with creations of Bell states $|\psi^B_{00}\rangle, |\psi^B_{01}\rangle, |\psi^B_{10}\rangle$ and $|\psi^B_{11}\rangle$, $(|0\rangle = |1\rangle_a|0\rangle_b, |1\rangle = |0\rangle_a|1\rangle_b)$.

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The state vector (7) comes at a controlled entrance of the *CNOT* gate. Thus, the ket-vector $|\varphi_1\rangle$ coming at the entrance of the *CNOT* gate can be represented in the form

$$|\varphi_1\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle_a|0\rangle_b + |0\rangle_a|1\rangle_b\right) \otimes \left(|1\rangle_a|0\rangle_b\right) = \frac{1}{\sqrt{2}}\left(|3\rangle_a|0\rangle_b + |1\rangle_a|2\rangle_b\right). \qquad (9)$$

Under the action of the unitary *CNOT* operator (for simplicity, denoted by U_{CN}) the ket-vector $|\varphi_1\rangle$ is transformed into one of the maximally entangled states (Bell states):

$$|\psi^B_{00}\rangle = U_{CN}|\varphi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} I & 0 \\ 0 & a^+b + ab^+ \end{pmatrix}\left(|3\rangle_a|0\rangle_b + |1\rangle_a|2\rangle_b\right) = \frac{1}{\sqrt{2}}\left(|3\rangle_a|0\rangle_b + |0\rangle_a|3\rangle_b\right). \qquad (10)$$

In the Eq. (10) the U_{CN} operator is determined by

$$U_{CN} = \begin{pmatrix} I & 0 \\ 0 & a^+b + ab^+ \end{pmatrix}, \qquad (11)$$

where I and 0 are the unit and zero operators defined in a two-dimensional Hilbert space.

Similarly, the other three Bell states $|\psi^B_{01}\rangle$, $|\psi^B_{10}\rangle$ and $|\psi^B_{11}\rangle$ can be obtained:

$$\begin{aligned} |\psi^B_{01}\rangle &= \frac{1}{\sqrt{2}}\left(|2\rangle_a|1\rangle_b + |1\rangle_a|2\rangle_b\right), \\ |\psi^B_{10}\rangle &= \frac{1}{\sqrt{2}}\left(|3\rangle_a|0\rangle_b - |0\rangle_a|3\rangle_b\right), \qquad (12) \\ |\psi^B_{11}\rangle &= \frac{1}{\sqrt{2}}\left(|2\rangle_a|1\rangle_b - |1\rangle_a|2\rangle_b\right). \end{aligned}$$

The Bell states $|\psi^B_{01}\rangle$, $|\psi^B_{10}\rangle$ and $|\psi^B_{11}\rangle$ from (12) are obtained using the same quantum circuit as in the case of the Bell state (10) by applying to the controlled and target entrances the basis vectors $|1\rangle_a|0\rangle_b$ and $|0\rangle_a|1\rangle_b$, $|0\rangle_a|1\rangle_b$ and $|1\rangle_a|0\rangle_b$, $|0\rangle_a|1\rangle_b$ and $|1\rangle_a|0\rangle_b$, respectively (Fig. 1, b - d).

The indices α and β for the Bell states $|\psi^B_{\alpha\beta}\rangle$ ($\alpha\beta$ = 00, 01, 10, 11) refer to the wave function of two qubits in the spinor representation, in which the spinors $|0\rangle$ and $|1\rangle$ are the basis functions of each of the qubits. In the two-boson representation, the states of coupled oscillators $|1\rangle_a|0\rangle_b$ and $|0\rangle_a|1\rangle_b$ correspond to these spinors. To simplify the notations, spinor notations are used for the indices α and β in the Bell states, as well as for the operators $L_{\alpha\beta}$ (see Section IV below), although all calculations were performed in the two-boson representation.

IV. Encoding the Information in a Quantum Computer Based on Bell States

Let us consider one of the Bell states, for example, $|\psi^B_{00}\rangle$. Let two scientists, Albert and Boris [10], are exchanging information via a quantum communication channel and are preparing several copies of the state $|\psi^B_{00}\rangle$. Boris takes one qubit from each pair. Albert encodes each pair of consecutive qubits $\alpha\beta$ by the operator $L_{\alpha\beta}$ and acts on his qubit of the pair by this operator. There are four $L_{\alpha\beta}$ operators: $L_{00} = I = a^+a + b^+b$, $L_{01} = \sigma_x = a^+b + ab^+$, $L_{10} = \sigma_z = a^+a - b^+b$, $L_{11} = i\sigma_y = a^+b - ab^+$, where σ_x, σ_y and σ_z are the Pauli matrices and **I** is the unit operator. Taking into consideration that $L_{\alpha\beta}$ operators have the properties

$$L_{00}|1\rangle_a|0\rangle_b = I|1\rangle_a|0\rangle_b = |1\rangle_a|0\rangle_b \,,$$
$$L_{00}|0\rangle_a|1\rangle_b = |0\rangle_a|1\rangle_b \,,$$
$$L_{01}|1\rangle_a|0\rangle_a = \sigma_x|1\rangle_a|0\rangle_a = |0\rangle_a|1\rangle_b \,,$$
$$L_{01}|0\rangle_a|1\rangle_b = \sigma_x|0\rangle_a|1\rangle_b = |1\rangle_a|0\rangle_b \,,$$
$$L_{10}|1\rangle_a|0\rangle_b = \sigma_z|1\rangle_a|0\rangle_b = |1\rangle_a|0\rangle_b \,,$$
$$L_{10}|0\rangle_a|1\rangle_b = \sigma_z|0\rangle_a|1\rangle_b = -|0\rangle_a|1\rangle_b \,,$$
$$L_{11}|1\rangle_a|0\rangle_b = i\sigma_y|1\rangle_a|0\rangle_b = -|0\rangle_a|1\rangle_b \,,$$
$$L_{11}|0\rangle_a|1\rangle_b = i\sigma_y|0\rangle_a|1\rangle_b = |1\rangle_a|0\rangle_b \,, \qquad (13)$$

we obtain all the four Bell states:

$$(L_{00}\otimes I\,)|\psi^B_{00}\rangle = \frac{1}{\sqrt{2}}(I\otimes I)\big(|3\rangle_a|0\rangle_b + |0\rangle_a|3\rangle_b\big) =$$
$$\tfrac{1}{\sqrt{2}}\big(|3\rangle_a|0\rangle_b + |0\rangle_a|3\rangle_b\big) = |\psi^B_{00}\rangle,$$
$$(L_{01}\otimes I\,)|\psi^B_{00}\rangle = \tfrac{1}{\sqrt{2}}(\sigma_x\otimes I)\big(|3\rangle_a|0\rangle_b + |0\rangle_a|3\rangle_b\big) =$$
$$\tfrac{1}{\sqrt{2}}\big(|2\rangle_a|1\rangle_b + |1\rangle_a|2\rangle_b\big) = |\psi^B_{01}\rangle,$$
$$(L_{10}\otimes I\,)|\psi^B_{00}\rangle = \tfrac{1}{\sqrt{2}}(\sigma_z\otimes I)\big(|3\rangle_a|0\rangle_b + |0\rangle_a|3\rangle_b\big) =$$
$$\tfrac{1}{\sqrt{2}}\big(|3\rangle_a|0\rangle_b - |0\rangle_a|3\rangle_b\big) = |\psi^B_{10}\rangle,$$
$$(L_{11}\otimes I\,)|\psi^B_{00}\rangle = \tfrac{1}{\sqrt{2}}\big(i\sigma_y\otimes I\big)\big(|3\rangle_a|0\rangle_b + |0\rangle_a|3\rangle_b\big) =$$
$$\tfrac{1}{\sqrt{2}}\big(|2\rangle_a|1\rangle_b - |1\rangle_a|2\rangle_b\big) = |\psi^B_{11}\rangle \qquad (14)$$

Eqs. (14) are particular cases of the following generalized equation:

$$\big(\,L_{\alpha\beta}\otimes I\big)|\psi^B_{00}\rangle = \big|\psi^B_{\alpha\beta}\big\rangle,\;\; \alpha,\beta = 0{,}1. \qquad (15)$$

To decode the information sent by Albert to Boris, it is necessary to start from the quantum circuit, by means of which the Bell states (10) and (12) were build:

$$\big|\psi^B_{\alpha\beta}\big\rangle = CNOT(H|\alpha\rangle)\otimes|\beta\rangle. \qquad (16)$$

Acting by the operator $(CNOT)^{-1}$ on the state vector $\big|\psi^B_{\alpha\beta}\big\rangle$, from Eq. (16), we can get:

$$(H|\alpha\rangle)\otimes|\beta\rangle = (CNOT)^{-1}\big|\psi^B_{\alpha\beta}\big\rangle. \qquad (17)$$

Operators *CNOT* and *H* have the properties

$$(CNOT)^2 = I, (CNOT)^{-1} = CNOT, \qquad (18)$$
$$H^2 = H.$$

This allows to perform the following decoding quantum circuit:

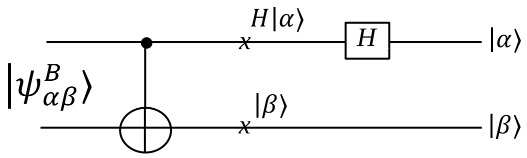

Fig. 1. The quantum circuit for decoding the information with transformation of Bell state $\big|\psi^B_{\alpha\beta}\big\rangle$ $(\alpha\beta = 00, 01, 10, 11)$ into spinor basis vectors $|\alpha\rangle$ and $|\beta\rangle$ $(\alpha\beta = 0{,}1)$; $(|0\rangle = |1\rangle_a|0\rangle_b, |1\rangle = |0\rangle_a|1\rangle_b)$.

Here is taken into account that the state vector H$|\alpha\rangle$ after passing through the Hadamard gate is transformed into a basis vector $|\alpha\rangle$ $(H\cdot H|\alpha\rangle = H^2|\alpha\rangle = |\alpha\rangle)$.

Conclusions

In quantum computer science, the formalism of spinor algebra is traditionally applied, along with other branches of mathematics, including: finding the eigenvalues and eigenfunctions of linear operators, finding the tensor products between state vectors, as well as between matrices, tensor products of linear spaces, and others.

This paper discusses the application of the Schwinger representation of the angular momentum to quantum computing. It is shown how to perform encoding and decoding of information in a quantum computer in this representation using the maximally entangled states (the Bell or EPR states).

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