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QUANTUM TREATMENT OF SELF-ORGANIZATION PHENOMENA OF EXCITONS AND BIEXCITONS

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In memory of Anatol Rotaru

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Abstract. We present in this paper the quantum treatment of self-organization phenomena of excitons and biexcitons in the geometry of a ring cavity. Applying the adiabatic elimination method of the exciton and biexciton variables, the Fokker-Planck equation for the transmitted field was obtained. The spectrum of transmission and second order correlation function were calculated. Thus, the spectrum of transmission describes a hysteresis cycle character, where a narrow spectral line in the end of cycle can be observed. The phenomena of optical bistability and switchings of excitons and biexcitons are investigated theoretically.

Keywords: *Optical bistability, semiconductors, Fokker-Planck Equation, self-organization, excitons, biexcitons, ring cavity.*

1. Introduction

During the last decade the phenomenon of optical bistability has been the subject of numerous theoretical and experimental investigations. It is one example of optical self-organisation of a system far from thermodynamic equilibrium. On the other hand, it opens up enormous opportunities for practical applications as optical logic device. A detailed classical description of optical bistability can be found in the monograph of Gibbs [1]. The first indication of optical bistability of excitons was done by Elesin and Kopaev [2]. The theoretical and experimental aspects Bose - Einstein condensation of excitons and biexitons are reported in [3]. The theory of stationary and dynamic optical bistability and self-pulsations of excitons and biexcitons in condensed media was elaborated in [4-6] and references are cited there. The prediction of optical bimodality induced by external additive noise with a finite bandwidth in the exciton-biexciton system is reported [7]. Theoretical investigation of controlling the optical bistability and optical multistability in a GaAs quantum well inside a unidirectional ring cavity is reported in [8].

In the last decade many studies implying self-organization effects of exciton systems were published. For exemple, a method of studying the dark dipolar excitons in transition metal dichalcogenide monolayers, considering a bilayer system of two-dimensional Bose-

Einstein-condensed dipolar dark excitons was proposed in [9]. It was demonstrated that interlayer interaction leads to a mixing state between excitations from different layers. An interesting idea to generate traveling pulses from an excitonic system created in a double quantum well heterostructure by a laser illumination is suggested in [10]. The dynamics of the excitons density for various illumination conditions is explored. Other model implying the formation of the excitonic condensed phase in quantum wells with defects of macroscopic size for planar quantum wells with various thickness was proposed recently by Sugakov [11]. The appearance of different types of structures in the exciton density distribution for large size of defects studied. New effects that appear at the control of exciton and self-organization of a quasi-two-dimensional nonequilibrium Bose-Einstein condensate in an in-plane potential were studied in [12, 13].

In this paper we propose a quantum treatment of the self-organization phenomena (optical bistability) of excitons and biexcitons of a semiconductor settled in a ring cavity with high quality factor Q.

2. Hamiltonian model, Fokker- Planck and Langevin equations

Our model consists of an ensemble of photons, excitons and biexcitons coupled to the thermostat and external field [7]. We consider in our analysis only one mode of excitons, biexcitons and photons. The full Hamiltonian of the systems, in the second order quantization, is given by

$$H = H_F + H_E + H_I + H_D, \tag{1}$$

where

$$H_F = \hbar\omega_1 a^+ a + \hbar\omega_2 b^+ b + \hbar\omega c^+ c,$$

$$H_E = i\hbar\left(Ec^+ e^{-i\omega_0 t} - E^* c e^{i\omega_0 t}\right),$$

$$H_I = -\hbar g\left(c^+ a + a^+ c\right) - \hbar\sigma g\left(a^+ b c^+ + c^- b^+ a\right),$$

$$H_D = \sum_{j=1}^{n}\left(\chi_1 a^+ \Gamma_{1j} + \chi_2 b^+ \Gamma_{2j} + \chi_3 c^+ \Gamma_{3j}\right) + c.c.$$

H_F represents the Hamiltonian of free excitons, biexcitons and photons, with $a\left(a^+\right)$, $b\left(b^+\right)$, $c\left(c^+\right)$ being the annihilation (creation) operators of excitons, biexcitons and photons, respectively. ω is the cavity mode frequency. $\hbar\omega_1$ ($\hbar\omega_2$) is the energy of exciton (biexciton) creation. H_E describes the Hamiltonian of interaction between the cavity field and external coherent field with amplitude E and frequency ω_0. H_I is the Hamiltonian of interaction between quasiparticles (photons, excitons and biexcitons). g is the constant of coupling between exciton and photon, and σ describes the conversion of excitons into biexcitons. H_D is the Hamiltonian of interaction of excitons, biexcitons and photons with the thermostats. The annihilation (creation) operators $\Gamma_{1j}\left(\Gamma_{1j}^+\right)$, $\Gamma_{2j}\left(\Gamma_{2j}^+\right)$, $\Gamma_{3j}\left(\Gamma_{3j}^+\right)$ correspond to excitonic, biexcitonic and photonic reservoirs, respectively while χ_1, χ_2, χ_3 are the coupling constants between the reservoirs and quasiparticles of the system.

From eq. (1) follows that the Hamiltonian H is time dependent. In order to eliminate this dependence, we use the rotating coordinate system with the frequency ω_0, that implies a new wave function $\psi = V\phi$. $V = e^{-i\omega_0 t N}$ is an unitary operator, and N represents the total number of quasi-particles. Thus, we can obtain an independent time Hamiltonian, where H_F and H_E parts of (1) have the following form:

$$H'_F = \hbar \cdot \Delta_1 \cdot a^+ a + \hbar \Delta_2 b^+ b + \hbar \Delta c^+ c\,, \quad H'_E = i\hbar\left(c^+ E - cE^*\right). \tag{2}$$

$\Delta_1 = \omega_1 - \omega_0$ is the detuning between the incident light (external) and exciton transition frequencies. $\Delta_2 = \omega_2 - 2\omega_0$ corresponds to the detuning between frequencies of the biexciton transition and incident light. On the other hand, $\Delta = \omega - \omega_0$ is the detuning between frequency of the cavity photons and incident light.

Eliminating the reservoir variables, we obtain the following master equation

$$\frac{d\rho}{dt} = \frac{1}{i\hbar}\left[H'_F + H'_E + H_I, \rho\right] + \left(\frac{\partial \rho}{\partial t}\right)_{ex-ph} + \left(\frac{\partial \rho}{\partial t}\right)_{biex-ph} + \left(\frac{\partial \rho}{\partial t}\right)_{f-d}, \tag{3}$$

Where

$$\left(\frac{\partial \rho}{\partial t}\right)_{ex-ph} = \gamma_1\left(1+\overline{n}_1\right)\left(\left[a\rho, a^+\right] + \left[a, \rho a^+\right]\right) + \gamma_1 \overline{n}_1\left(\left[a^+\rho, a\right] + \left[a^+, \rho a\right]\right),$$

$$\left(\frac{\partial \rho}{\partial t}\right)_{biex-ph} = \gamma_2\left(1+\overline{n}_2\right)\left(\left[b\rho, b^+\right] + \left[b, \rho b^+\right]\right) + \gamma_2 \overline{n}_2\left(\left[a^+\rho, a\right] + \left[b^+, \rho b\right]\right),$$

$$\left(\frac{\partial \rho}{\partial t}\right)_{f-d} = \kappa\left(1+\overline{n}\right)\left(\left[c\rho, c^+\right] + \left[c, \rho c^+\right]\right) + \kappa \overline{n}\left(\left[c^+\rho, c\right] + \left[c^+, \rho c\right]\right).$$

The parameters γ_1, γ_2, κ represent the amortization rates of the excitons, biexcitons and photons, respectively ($\gamma_1 = \pi\left|\chi_1\right|^2$, $\gamma_2 = \pi\left|\chi_2\right|^2$, $\kappa = \pi\left|\chi_3\right|^2$). $\overline{n}_i = \dfrac{1}{\exp\left(\dfrac{\hbar\omega_i}{kT}\right) - 1}$ is the average value of the thermal particles of the reservoirs at temperature *T*.

Using the generalized non-diagonal *p* representation Drummond-Gardiner [14, 15], the master equation (3) can be transformed in the Fokker-Planck equation. The complex space of representation is generated defining the correspondence between the operators and complex parts as follow: $a \to \alpha_1,\, b \to \alpha_2,\, c \to \alpha_3;\, a^+ \to \beta_1,\, b^+ \to \beta_2,\, c^+ \to \beta_3$. The statistic operator $\hat{\rho}$ can be associated with a complex distribution function *p*, via the relation

$$\rho(f) = \int_l \int_{l'} p\left(\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3\right) \lambda\left(\alpha_i, \beta_i\right) d\mu, \tag{4}$$

where $\lambda\left(\alpha_i, \beta_i\right) = \dfrac{\left|\alpha\right\rangle\left\langle\beta^*\right|}{\left\langle\beta^*\middle|\alpha\right\rangle}$ is the operator of projection and $d\mu = d\alpha_1\, d\alpha_2\, d\alpha_3\, d\beta_1\, d\beta_2\, d\beta_3$ represents the measure of integration of domain *D*. It worth to mention, that the contours of integration l and l' are independent. The Fokker-Planck equation can be written from master equation using the relations of the following correspondence

$$a\rho \to \alpha \;\; P(\vec{\alpha}); \qquad \rho a \to \left(\alpha + \frac{\partial}{\partial \beta}\right) P(\vec{\alpha});$$
$$a^{+}\rho \to \left(\beta + \frac{\partial}{\partial \alpha}\right) P(\vec{\alpha}); \quad \rho a^{+} \to \alpha \;\; P(\vec{\alpha}). \tag{5}$$

Using expressions (5) we obtain the new form of Fokker- Planck equation

$$\frac{\partial P(\alpha_i,\beta_i,t)}{\partial t} = \left\{ -\frac{\partial}{\partial \alpha_1}\left(-i\Delta_1\alpha_1 + ig\alpha_3 + i\sigma g\beta_3\alpha_2 - \gamma_1\alpha_1\right) - \right.$$
$$-\frac{\partial}{\partial \alpha_2}\left(-i\Delta_2\alpha_2 + i\sigma g\alpha_1\alpha_3 - \gamma_2\alpha_2\right) - \frac{\partial}{\partial \alpha_3}(-i\Delta\alpha_3 + ig\alpha_1 +$$
$$+i\sigma g\beta_1\alpha_2 - \kappa\alpha_3 + E) - \frac{\partial}{\partial \beta_1}\left(i\Delta_1\beta_1 - ig\beta_3 - i\sigma g\alpha_3\beta_2 - \gamma_1\beta_1\right) -$$
$$-\frac{\partial}{\partial \beta_2}\left(i\Delta_2\beta_2 - i\sigma g\beta_1\beta_2 - \gamma_2\beta_2\right) - \frac{\partial}{\partial \beta_3}(i\Delta\beta_3 + E^{*} - ig\beta_1 -$$
$$-i\sigma g\alpha_1\beta_2 - \kappa\beta_3) + \frac{\partial^2}{\partial \alpha_1 \partial \alpha_3}\left(i\sigma g\alpha_2\right) + \frac{\partial^2}{\partial \beta_1 \partial \beta_3}\left(-i\sigma g\beta_2\right) +$$
$$\left. + \frac{\partial^2}{\partial \alpha_1 \partial \beta_1}\left(2\gamma_2\bar{n}_2\right) + \frac{\partial^2}{\partial \alpha_3 \partial \beta_3}\left(2\kappa\bar{n}\right) \right\} P. \tag{6}$$

It is well known that each Fokker - Planck process can be associated with a system of stochastical differential equations called Langevin equations

$$\frac{d\alpha_1}{dt} = -i\Delta_1\alpha_1 + ig\alpha_3 + i\sigma g\beta_3\alpha_2 - \gamma_1\alpha_1 + \Gamma_{\alpha_1}(t),$$
$$\frac{d\alpha_2}{dt} = -i\Delta_2\alpha_2 + i\sigma g\alpha_1\alpha_3 - \gamma_2\alpha_2 + \Gamma_{\alpha_2}(t),$$
$$\frac{d\alpha_3}{dt} = E - i\Delta\alpha_3 - \kappa\alpha_3 + ig\alpha_1 + i\sigma g\beta_1\alpha_2 + \Gamma_{\alpha_3}(t),$$
$$\frac{d\beta_1}{dt} = i\Delta_1\beta_1 - ig\beta_3 - i\sigma g\alpha_3\beta_2 - \gamma_1\beta_1 + \Gamma_{\beta_1}(t), \tag{7}$$
$$\frac{d\beta_2}{dt} = i\Delta_2\beta_2 - i\sigma g\beta_1\beta_3 - \gamma_2\beta_2 + \Gamma_{\beta_2}(t),$$
$$\frac{d\beta_3}{dt} = E^{*} + i\Delta\beta_3 - ig\beta_1 - i\sigma g\alpha_1\beta_3 - \kappa\beta_3 + \Gamma_{\beta_3}(t),$$

where the stochastical Langevin forces are connected with the coefficients of diffusion of Fokker-Planck equation through the correlation relations:

$$\left\langle \Gamma_{\alpha_1}(t)\Gamma_{\alpha_3}(t)\right\rangle = i\sigma g\alpha_2\delta(t-t'), \tag{8}$$

$$\left\langle \Gamma_{\beta_1}(t)\Gamma_{\beta_3}(t')\right\rangle = -i\sigma g\beta_2\delta(t-t') \tag{9}$$

$$\left\langle \Gamma_{\alpha_1}(t)\Gamma_{\beta_1}(t')\right\rangle = 2\gamma_1\bar{n}_1\delta(t-t') \tag{10}$$

$$\left\langle \Gamma_{\alpha_2}(t)\Gamma_{\beta_2}(t')\right\rangle = 2\gamma_1\bar{n}_2\delta(t-t') \tag{11}$$

$$\left\langle \Gamma_{\alpha_3}(t)\,\Gamma_{\beta_3}(t')\right\rangle = 2\kappa\bar{n}\delta(t-t') \tag{12}$$

Considering γ_1 and γ_2 much higher than κ, so that the exciton and biexciton variables can be neglected, we obtain a system of stochastical differential equation: $\frac{\partial\alpha_1}{\partial t} = \frac{\partial\alpha_2}{\partial t} = \frac{\partial\beta_1}{\partial t} = \frac{\partial\beta_2}{\partial t} = 0$. In the exact resonance case, i.e., $\Delta_1 = \Delta_2 = \Delta = 0$, the relations for exciton and biexciton variables have the following form:

$$\alpha_1 = \frac{ig\alpha_3}{M\gamma_1} + \frac{i\sigma g\beta_3}{M\gamma_1\gamma_2}\Gamma_{\alpha_2}(t) + \frac{1}{\gamma_1 M}\Gamma_{\alpha_1}(t), \tag{13}$$

$$\alpha_2 = -\frac{\sigma g^2\alpha_3^2}{\gamma_1\gamma_2 M} + \frac{i\sigma g\alpha_3}{M\gamma_1\gamma_2}\Gamma_{\alpha_1}(t) - \left[\frac{\sigma^2 g^2\alpha_3\beta_3}{M\gamma_1\gamma_2^2} - \frac{1}{\gamma_2}\right]\Gamma_{\alpha_2}(t), \tag{14}$$

$$\beta_1 = -\frac{ig\beta_3}{M\gamma_1} - \frac{i\sigma g\alpha_3}{M\gamma_1\gamma_2}\Gamma_{\beta_2}(t) + \frac{1}{\gamma_1 M}\Gamma_{\beta_1}(t), \tag{15}$$

$$\beta_2 = -\frac{\sigma g^2\beta_3^2}{\gamma_1\gamma_2 M} - \left[\frac{\sigma^2 g^2\alpha_3\beta_3}{M\gamma_1\gamma_2^2} - \frac{1}{\gamma_2}\right]\Gamma_{\beta_2}(t) - \frac{i\sigma g\beta_3}{M\gamma_1\gamma_2}\Gamma_{\beta_1}(t), \tag{16}$$

where $M = 1 + \frac{\sigma^2 g^2\alpha_3\beta_3}{\gamma_1\gamma_2 M}$.

Introducing the equations (13) – (16) into (8) – (9), one can obtain the correlation expressions that depend only on the stochastic terms. In what follows we approximate the expressions for α_2 and β_2 with the deterministic stationary relations

$$\alpha_2 \cong -\frac{\sigma g^2\alpha_3^2}{\gamma_1\gamma_2 M}; \quad \beta_2 \cong -\frac{\sigma g^2\beta_3^2}{\gamma_1\gamma_2 M}. \tag{17}$$

After elimination of the exciton and biexciton variables, the correlation relations (8)–(9) can be written in the following form

$$\left\langle \Gamma_{\alpha_1}(t)\Gamma_{\alpha_3}(t')\right\rangle = -\frac{i\sigma^2 g^3\alpha_3^2}{\gamma_1\gamma_2 M}\delta(t-t'), \tag{18}$$

$$\left\langle \Gamma_{\beta_1}(t)\Gamma_{\beta_3}(t')\right\rangle = \frac{i\sigma^2 g^3\beta_3^2}{\gamma_1\gamma_2 M}\delta(t-t'). \tag{19}$$

We calculate the drift coefficients introducing eqs. (13)–(16) into (7). Finally, we obtain the Langevin equations for the field

$$\frac{d\alpha_3}{dt}=E-\kappa\alpha_3-\frac{g^2\alpha_3}{M\gamma_1}-\frac{g^4\sigma^2\alpha_3^2\beta_3}{\gamma_1^2\gamma_2M^2}-\frac{4\sigma^6g^8\alpha_3^3\beta_3^2}{\gamma_1^4\gamma_1^3M^4}\Gamma_\alpha(t), \quad (20)$$

$$\frac{d\beta_3}{dt}=E^*-\kappa\beta_3-\frac{g^2\beta_3}{M\gamma_1}-\frac{g^4\sigma^2\alpha_3\beta_3^2}{\gamma_1^2\gamma_2M^2}-\frac{4\sigma^6g^8\alpha_3^2\beta_3^3}{\gamma_1^4\lambda_2^3M^4}+\Gamma_\beta(t), \quad (21)$$

where the new stochastic terms have the form

$$\Gamma_\alpha(t)=A\Gamma_{\alpha_1}(t)+B\Gamma_{\alpha_2}(t)+C\Gamma_{\beta_1}(t)+D\Gamma_{\beta_2}(t)+\Gamma_{\alpha_3}(t), \quad (22)$$

$$\Gamma_\beta(t)=C^*\Gamma_{\alpha_1}(t)+D^*\Gamma_{\alpha_1}(t)+A^*\Gamma_{\beta_1}(t)+B^*\Gamma_{\beta_2}(t)+\Gamma_{\beta_3}(t), \quad (23)$$

And

$$\begin{aligned} A&=\frac{ig}{\gamma_1M}+\frac{i\sigma^2g^2\alpha_3\beta_3}{M^2\gamma_1^2\gamma_2},\\ B&=\frac{-\sigma^3g^4\alpha_3\beta_3^2}{\gamma_1^2\gamma_2^2M^2},\\ C&=-\frac{i\sigma^2g^3\alpha_3^2}{\gamma_1^2M^2\gamma_2},\\ D&=-\frac{\sigma^3g^3\alpha_3^3}{M^2\gamma_1^2\gamma_2^2}. \end{aligned} \quad (24)$$

Thus, the correlations of the new stochastical terms (22) – (23) can be written in the new form

$$\langle\Gamma_\alpha(t)\Gamma_\alpha(t')\rangle=D_{\alpha\alpha}\delta(t-t')=\left[\frac{2\sigma^2g^4\alpha_3^2}{\gamma_1^2\gamma_2M^2}\left(1+\frac{\sigma^2g^2\alpha_3\beta_3}{M\gamma_1\gamma_2}\right)\right]\delta(t-t'), \quad (25)$$

$$\langle\Gamma_\alpha(t)\Gamma_\beta(t')\rangle=D_{\alpha\beta}\delta(t-t')=\frac{2\sigma^4g^6\alpha_3^2\beta_3^2}{\gamma_1^3\gamma_2^2M^3}\delta(t-t'). \quad (26)$$

In the case of very low temperature of reservoirs, we can neglect in expressions for stochastical correlation (14)-(15) the terms proportional to $\bar{n}_1,\bar{n}_2,\bar{n}$, being very small i.e. $\kappa T<<\hbar\omega_i$. We are interested only in the quantum fluctuations that appear at non-linear interaction of excitons and biexcitons. Thus, the thermal fluctuations (fluctuations due to the reservoir interaction) could be neglected. Taking into account the equivalence between the Langevin and Fokker-Planck equations, a new Fokker-Planck relation for field behavior can be obtained

$$\frac{\partial P(\alpha_3,\beta_3,t)}{\partial t}=-\frac{\partial}{\partial\alpha_3}\left[\left(E-\kappa\alpha_3-\frac{g^2\alpha_3}{M\gamma_1}-\frac{g^4\sigma^2\alpha_3^2\beta_3}{\gamma_1^2\gamma_2M^2}-\right.\right.$$

$$-\frac{4\sigma^6 g^8 \alpha_3^3 \beta_3^2}{\gamma_1^4 \gamma_2^3 M^4}\Bigg) P\Bigg] - \frac{\partial}{\partial \beta_3}\Bigg[\Bigg(E^* - \kappa\beta_3 - \frac{g^2 \beta_3}{M\gamma_1} - \frac{g^4 \sigma^2 \alpha_3 \beta_3^2}{\gamma_1^2 \gamma_2 M^2} -$$

$$-\frac{4\sigma^6 g^8 \alpha_3^3 \beta_3^2}{\gamma_1^4 \gamma_2^3 M^4}\Bigg) P\Bigg] + \frac{1}{2}\frac{\partial^2}{\partial \alpha_3^2}\left\{\left[\frac{2\sigma^2 g^4 \alpha_3^2}{\gamma_1^2 \gamma_2 M^2}\left(1 + \frac{\sigma^2 g^2 \alpha_3 \beta_3}{M\gamma_1\gamma_2}\right)\right] P\right\} +$$

$$+\frac{1}{2}\frac{\partial^2}{\partial \beta_3^2}\left\{\left[\frac{2\sigma^2 g^4 \beta_3^2}{\gamma_1^2 \gamma_2 M^2}\left(1 + \frac{\sigma^2 g^2 \alpha_3 \beta_3}{M\gamma_1\gamma_2}\right)\right] P\right\} + \frac{\partial^2}{\partial \alpha_3 \partial \beta_3}\left[\frac{2\sigma^4 g^6 \alpha_3^2 \beta_3^2}{\gamma_1^3 \gamma_2^2 M^3} P\right]. \quad (27)$$

In what follows we are interested in the behavior of transmitted light amplitude. Thus, it is more convenient to consider the polar coordinates

$$\alpha_3 = r e^{-i\varphi}, \quad r^2 = \alpha_3 \beta_3,$$
$$\beta_3 = r e^{i\varphi}, \quad \varphi = \frac{1}{2i}\ln\frac{\beta_3}{\alpha_3}. \quad (28)$$

The general expressions of drift and diffusion coefficients of Fokker-Planck equation [16] are given by

$$a_i' = a_i'(a_i, t),$$
$$A_i = \frac{\partial a_i'}{\partial t} + \left(\frac{\partial a_i'}{\partial a_\kappa}\right) A_\kappa + \frac{\partial^2 a_i'}{\partial a_m \partial a_n} D_{mn}, \quad (29)$$
$$D_{ij} = \frac{\partial a_i'}{\partial a_\kappa}\frac{\partial aj'}{\partial a_l} D_{\kappa l}.$$

Taking into account the equations (28) and (29) it become easy to obtain the phase and amplitude coefficients of Fokker-Planck equation

$$A_r = r\,\mathrm{Re}\left(\frac{A_{\alpha_3}}{\alpha_3}\right) - \frac{r}{2}\mathrm{Re}\left(\frac{D_{\alpha\alpha}}{\alpha_3^2}\right) + \frac{1}{2r} D_{\alpha\beta},$$

$$A_\varphi = -\mathrm{Im}\left(\frac{A_{\alpha_3}}{\alpha_3} - \frac{D_{\alpha\alpha}}{\alpha_3^2}\right),$$

$$D_{rr} = \frac{r^2}{2}\mathrm{Re}\left(\frac{D_{rr}}{\alpha_3^2}\right) + \frac{1}{2} D_{\alpha\beta}, \quad (30)$$

$$D_{\varphi\varphi} = \frac{1}{2r^2} D_{\alpha\beta} - \frac{1}{2}\mathrm{Re}\left(\frac{D_{\alpha\alpha}}{\alpha_3^2}\right),$$

$$D_{r\varphi} = -\frac{r}{2}\mathrm{Im}\left(\frac{D_{\alpha\alpha}}{\alpha_3^2}\right),$$

where A_{α_3}, A_{β_3}, $D_{\alpha\alpha}$, $D_{\alpha\beta}$ are the drift and diffusion terms of Fokker – Plank equation (27). Substituting these coefficients into (28) we obtain the Fokker-Planck equation as function of variable χ and φ

$$\frac{\partial P(x_1\varphi_1 t)}{\partial\tau} = \frac{\partial}{\partial x}\left\{\left[x\left(1+\frac{2c(1+2x^2)}{(1+x^2)^2}+\frac{2(1+2x^2+5x^4)}{(1+x^2)^4}\right)-y\cos\varphi\right]P\right\}+$$

$$+\frac{\partial}{\partial\varphi}\left(\frac{y}{x}\sin(\varphi)P\right)+\frac{2}{2}\frac{\partial^2}{\partial x^2}\left[\frac{x^2(1+3x^2)}{(1+x^2)^3}P\right]+\frac{2}{2}\frac{\partial^2}{\partial\varphi^2}\left[\frac{1}{(1+x^2)^3}P\right], \qquad (31)$$

where $x = \frac{r}{\sqrt{\pi_0}}$ and $y = \frac{E}{\kappa\sqrt{n_0}}$ describe the normalized amplitude of transmitted and incident light, respectively. $\tau = kt$ is the normalized time and $C = \frac{g^2}{2\gamma_1\kappa}$ is a constant. $q = \frac{2C}{n_0}$ represents the parameter that describes the quantum fluctuations and $n_0 = \frac{\gamma_1\gamma_2}{\sigma^2 g^2}$.

Neglecting the phase variation, one can obtain the Fokker-Planck equation only for amplitude:

$$\frac{\partial P}{\partial t}+\frac{\partial J}{\partial x}=0, \qquad (32)$$

where *J* is the density flow with the form

$$J = -\left\{x\left[1+\frac{2c(1+2x^2)}{(1+x^2)^2}+\frac{2(1+2x^2+5x^4)}{(1+x^2)^4}\right]-y\right\}P-\frac{q}{2}\frac{\partial}{\partial x}\left[\frac{x^2(1+3x^2)}{(1+x^2)^3}P\right]. \qquad (34)$$

We have to check the balance condition at the steady state case considering

$$J = 0. \qquad (35)$$

The equation (35) represents a first order differential equation, where the density of probability $P(x)$ is an unknown function and has the following solution

$$P(x) = N\exp\left[-\frac{2}{q}\phi(x)\right], \qquad (36)$$

where N is the normalized constant and $\phi(x)$ describes the potential function that is given by

$$\phi(x) = \int\left\{y-x\left[1+\frac{2c(1+2x^2)}{(1+x^2)^2}+\frac{q(1+2x^2+5x^4)}{(1+x^2)^4}\right]\right\}\bullet\frac{(1+x^2)^3}{x^2(1+3x^2)}dx-\frac{q}{2}\ln\left[\frac{x^2(1+3x^2)}{(1+x^2)^3}\right]. \qquad (37)$$

On the other hand, in the deterministic case, the steady states of a system that implies quantum fluctuations, characterize points for which the density of probability take extreme values. For function $P(x)$, the maximum of these points corresponds to the most probable states, while the minimum corresponds to less probable states. The extremes of function $\phi(n)$ coincide with those of $P(x)$, i.e. the points for which $P(x)$ is maximal correspond to the points with minimal values for $\phi(x)$, and vice-versa. It is known that the steady states of a system are given by the condition $\phi'(x)=0$, which implies

$$y = f(x,c,q), \tag{38}$$

where

$$f(x,c,q) = x\left(1+\frac{2c\left(1+2x^2\right)}{\left(1+x^2\right)^2}\right)+\frac{2qx\left(1+3x^2+x^4\right)}{\left(1+x^2\right)^4}.$$

Following the assumption [7], we determine the diagram *q* as function of new parameter *C*, that describes a behavior $f(x,q,C)$. The critical values of *q* and *C* can be obtained from conditions $f'_x = f''_{xx} = 0,$ which are equivalent to the next system of equations

$$\begin{aligned} &1-2c\psi(z,k)=0 \\ &\psi'(z,k)=0, \end{aligned} \tag{39}$$

where $z \equiv x^2,\ k=\dfrac{q}{C}$

$$\psi(z,k)=\frac{2z^4+(3k+1)z^3+(10k-5)z^2-(2k+5)z-(k+1)}{\left(1+z^5\right)} \tag{40}$$

$$\psi'(z,k)=-\frac{2z^4+(6k-6)z^3+(21k-18)z^2-(28k+10)z-3k}{(1+z)^6}. \tag{41}$$

From equations (39) follows the relations for parameters *C* and *q* [7]

$$\begin{aligned} C &= \frac{(1+z)^5\left(6z^3+21z^2-28z-3\right)}{2\left(6z^7+46z^6+53z^5+31z^4+84z^3+96z^2+33z+3\right)} \\ q &= \frac{(1+z)^5\left(-2z^4+6z^3+18z^2+10z\right)}{2\left(6z^7+46z^6+53z^5+31z^4+84z^3+96z^2+33z+3\right)}. \end{aligned} \tag{42}$$

The system (42) represents the parametric equation of separatrice in the space (C, q). This separatrice is plotted in the Figure 1. As one can see, the domain of parameters *q* and *C* is separated into two regions. In the region I the system is mono-stable. The region II is characterized by bistable states. We mention that, the critical value $C=\frac{54}{17}$ corresponds to the deterministic case (see point A in Figure 1).

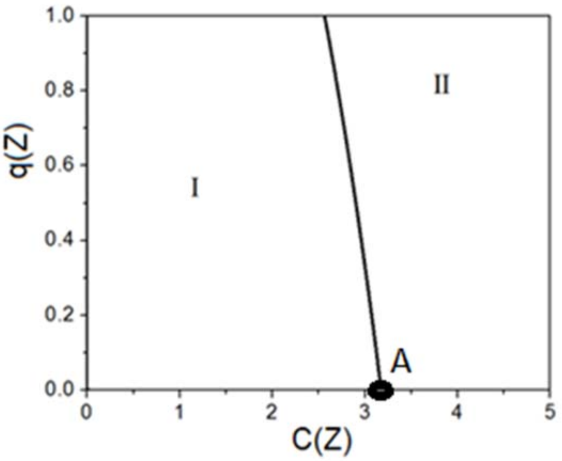

Figure 1. The separatrice line in space (q,C).

3. Fluctuation evaluation. Optical bistability.

In what follows, we analyze the case of small fluctuations for the transmitted field through the semiconductor. The correlations and spectral line expressions of transmitted radiation can be evaluated via the linearization of stochastical differential equations around the stable steady states. This type of linearization transforms the stochastical equations in those of Ornstein-Uhlenbeck, which solutions are known and allow easy to calculate all characteristic values of fluctuations. Applying the linearization procedure of stochastical differential equations one can obtain the system:

$$\frac{\partial}{\partial t}\left[\delta\vec{\alpha}_{\mu}(t)\right]=-\left[\frac{\partial}{\partial\alpha_{\nu}}\overline{A}_{\mu}(\vec{\alpha}_0)\right]\delta\vec{\alpha}_{\nu}+\left[D(\vec{\alpha}_0)\right]^{\frac{1}{2}}_{\mu\nu}\xi(t), \quad (43)$$

where $A_{\mu\nu}=\frac{\partial\overline{A}_{\mu}(\vec{\alpha}_0)}{\partial\alpha_{\nu}}$ characterize the coefficients of linearization associated with the drift and $D(\vec{\alpha}_0)$ is the matrix of diffusion. Both characteristics, i.e., the diffusion and drift are evaluated in the point $\vec{\alpha}_0\left[\vec{\alpha}=(\alpha_3,\beta_3)\right]$, and describe the solution of equation $\vec{A}_{\mu}(\vec{\alpha}_0)=0$. From the system of equation $\overline{A}_{\mu}(\vec{\alpha}_0)=0$ we obtain the curve of steady states plotted in Figure 2.

$$y=x_0\left(1+\frac{2C\left(1+2x_0^2\right)}{\left(1+x_0^2\right)^2}\right]. \quad (44)$$

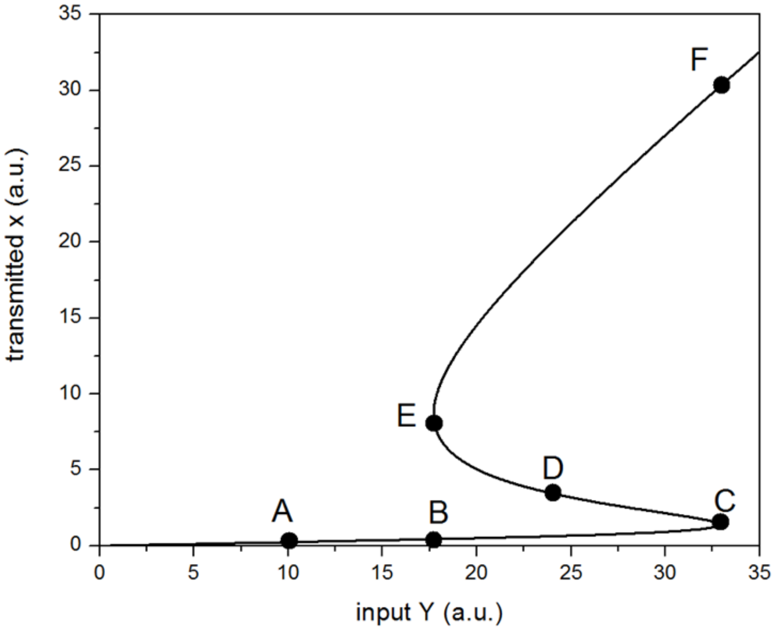

Figure 2. Stationary curve of bistability calculated from relation (44).

The drift and diffusion matrixes have the form

$$A_{\mu\nu} = \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix}, \quad D_{\mu\nu} = \begin{pmatrix} -d & \lambda \\ \lambda & -d^* \end{pmatrix}, \tag{45}$$

where a, b, d, λ are given by

$$a = k\left[1 + \frac{2c\left(1+3x_0^2\right)}{\left(1+x_0^2\right)^3} + \frac{4qx_0^4\left(3-x_0^2\right)}{\left(1+x_0^2\right)^5}\right],$$

$$b = k\left[-\frac{4x_0^2 C}{\left(1+x_0^2\right)^3} + \frac{8qx_0^2\left(1-x_0^2\right)}{\left(1+x_0^2\right)^5}\right]\eta_s^{\ 2},$$

$$d = -\frac{4Ck\left(1+2x_0^2\right)\eta_s^{\ 2}}{\left(1+x_0^2\right)^3}, \quad \lambda = \frac{4Ck\, x_0^4}{\left(1+x_0^2\right)^3}. \tag{46}$$

The parameters $\eta_s = \dfrac{\alpha_{3s}}{\sqrt{n_0}}$; $x_0^2 = \eta_s \eta_s^*$, x_0 satisfy the relation (44). We consider an Ornstein-Uhlenbeck process. Thus, for the correlation matrix [6] the following expression can be used:

$$C_{\mu\nu} = \left\langle \delta\alpha_\mu \delta\alpha_\nu \right\rangle = \begin{pmatrix} \left\langle \delta\alpha_3\, \delta\alpha_3 \right\rangle & \left\langle \delta\alpha_3 \delta\beta_3 \right\rangle \\ \left\langle \delta\beta_3 \delta\alpha_3 \right\rangle & \left\langle \delta\beta_3 \delta\beta_3 \right\rangle \end{pmatrix} = \\ = \frac{D \cdot \det A + \left[A - \mathrm{I}\cdot \mathrm{Tr}(A)\right] D \left[A^T - \mathrm{I}\cdot \mathrm{Tr}(A)\right]}{2\cdot \mathrm{Tr}(A) \det(A)}. \tag{47}$$

Introducing the expression (45) and (46) into (47) we obtain the following correlation matrix

$$C_{\mu\nu}=\frac{1}{4Re(a)\lambda_0}\begin{pmatrix}-d\left(\left(a^*\right)^2+\lambda_0\right)-d^*b^2-2a^*b\Gamma & 2\left[\lambda|a|^2+Re\left(abd^*\right)\right]\\ 2\left[\lambda|a|^2+Re\left(abd^*\right)\right] & -d^*\left(a^2+\lambda_0\right)-db^{*2}-2ab^*\Gamma\end{pmatrix}, \quad (48)$$

where $\lambda_0=|a|^2-|b|^2$. Using the matrix $C_{\mu\nu}$ it become possible to calculate the relative values of fluctuation intensity $\frac{\delta \mathrm{I}}{\mathrm{I}_{coh}}$, as well as the normalized correlation function of the second order $g^2(0)$. The relative value of intensity of fluctuations is defined as the ratio between the fluctuation intensity $\delta\mathrm{I}_{inch}=\langle\delta\beta_3\ \delta\alpha_3\rangle$ and the intensity of coherent light $\mathrm{I}_{coh}=n_0x_0^2$

$$\frac{\delta\mathrm{I}_{incoh}}{\mathrm{I}_{coh}}=\frac{\lambda|a|^2+Re\left(abd^*\right)}{2a_R\lambda_0n_0x_0^2}. \quad (49)$$

We introduce (46) in (49) and obtain

$$\frac{\delta\,\mathrm{I}_{inch}}{\mathrm{I}_{coh}}=\frac{2ck^2x_0^2}{n_0\lambda_0\left(x_0\right)\left(1+x_0^2\right)^3}\left[1+\frac{2c\left(1+5x_0^2+4x_0^4\right)}{\left(1+x_0^2\right)^3}\right], \quad (50)$$

where

$$\lambda_0\left(x_0\right)=k^2\left\{\left[1+\frac{2C\left(1+3x_0^2\right)}{\left(1+x_0^2\right)^3}\right]^2-\frac{16C^2x_0^8}{\left(1+x_0^2\right)^6}\right\}.$$

Assuming $(q<<1)$, in coefficients *a* and *b*, the terms proportional to parameter *q* can be neglected. The same approximation will be considered and in the following calculations.

The second order correlation function it is calculated using the relation

$$g^{(2)}\left(0\right)=1+\frac{2}{\mathrm{I}_{coh}}\left[\langle\delta\beta_3\ \delta\alpha_3\rangle+R_e\left(\frac{\beta_3}{\alpha_3}\langle\delta\alpha_3\ \delta\alpha_3\rangle\right)\right]. \quad (51)$$

Using the correlation matrix (48) we obtain

$$g^{(2)}\left(0\right)=1+\frac{2\lambda|a|^2+Re\left\{2abd^*-\frac{\eta_s^*}{\eta_s}\left[d^2b^*+d\left(a^2+\lambda_0\right)+2ab\lambda\right]\right\}}{2x_0^2Re\left(a\right)\lambda_0\left(x_0\right)\cdot n_0}. \quad (52)$$

Finally, we replace the coefficients (46) in (52)

$$g^{(2)}\left(0\right)=1+\frac{4ck^2}{\left(1+x_0^2\right)^3\lambda_0\left(x_0\right)n_0}\left[1+3x_0^2+\frac{2C\left(1+6x_0^2+11x_0^4+6x_0^6\right)}{\left(1+x_0^2\right)^3}\right]. \quad (53)$$

The spectrum $S(\omega)$ of the light that cross a crystal is proportional to the Fourier transform of the autocorrelation function $\langle\beta_3(t)\ \alpha_3(0)\rangle$, and it is composed by the coherent and incoherent parts

$$S(\omega)=S_{coh}(\omega)+S_{incoh}(\omega). \tag{54}$$

The coherent spectral part of transmitted light $S_{coh}(\omega)=n_0x_0^2\delta(\omega-\omega_0)$ is just a function of the incident light frequency ω_0, while the incoherent component of the spectrum is given by [14]

$$S_{incoh}(\omega)=\frac{1}{2\pi}\left[\left(A+i\omega\mathrm{I}\right)^{-1}D\left(A^T-i\omega\mathrm{I}\right)^{-1}\right]. \tag{55}$$

We are interested only in the incoherent component of the spectrum and it can be obtained by introducing (45) in (55)

$$S_{incoh}(\omega)=\frac{\gamma\lambda\omega^2+2\omega\,Im\left(b^*d+\lambda a\right)+\lambda\left(|a|^2+|b|^2+2Re\left(abd^*\right)\right)}{2\pi\,\lambda(\omega)}, \tag{56}$$

where $\lambda(\omega)=\omega^4+\omega^2\left(a^2+a^{*2}+2|b|^2\right)+\left(|a|^2-|b|^2\right)^2$.

Taking into account the parameters a,b,d,λ we get the next relation for the spectrum

$$S_{incoh}(\omega)=\frac{\lambda(x_0)\omega^2+P(x_0)}{2\pi\left[\omega^4+\lambda_1(x_0)\omega^2+\lambda_2(x_0)\right]}, \tag{57}$$

where

$$P(x_0)=\frac{4ck^3x_0^4}{\left(1+x_0^2\right)^3}\left\{\left[1+\frac{2c\left(1+3x_0^2\right)}{\left(1+x_0^2\right)^3}\right]\left[1+\frac{2c\left(1+7x_0^2+8x_0^4\right)}{\left(1+x_0^2\right)^3}\right]+\frac{16c^2x_0^8}{\left(1+x_0^2\right)^6}\right\},$$

$$\lambda_1(x_0)=2k^2\left\{\left[1+\frac{2C\left(1+3x_0^2\right)}{\left(1+x_0^2\right)^3}\right]^2+\frac{16C^2x_0^8}{\left(1+x_0^2\right)^6}\right\},$$

$$\lambda_2(x_0)=k^4\left\{\left[1+\frac{2C\left(1+3x_0^2\right)}{\left(1+x_0^2\right)^3}\right]^2-\frac{16C^2x_0^8}{\left(1+x_0^2\right)^6}\right\}^2.$$

The plots of the spectral fluctuations for different values of x_0 from the optical bistability curve shown in Figure 2, are represented in Figure 3.

Each plot of Figure 3 corresponds to the marked point (A-F) of optical bistability curve shown in Figure 2. From these pictures we can observe a hysteresis cycle of spectrum, where the extreme regions are marked by two extremely narrow spectral lines for conditions of points C and E in Figure 2. These situations correspond to the jumped points from one region (branch) of the stationary curve of optical bystability to another, as shown in Figure 3 (c) and (e). On the lower branch of optical bistability curve (small values of x_0) the spectrum width is high see Figure 3 (a). When increasing parameter x_0 (see Figure 3 b) the line width become narrow, achieving a critical value at the jumped point C of Figure 2. On the uppers branch of optical bistability curve, the spectrum width is narrow (see Figure 3 (d)) and continues to diminish when x_0 is moving to the next jumped point E. Thus, based on the quantum description, we demonstrated in this paper the presence of optical bistability for a system of excitons and biexcitons with a geometry of a ring resonator with high quality factor Q. Finally, it worth to mention that the obtained results are in a good agreement with those obtained by in [17, 18].

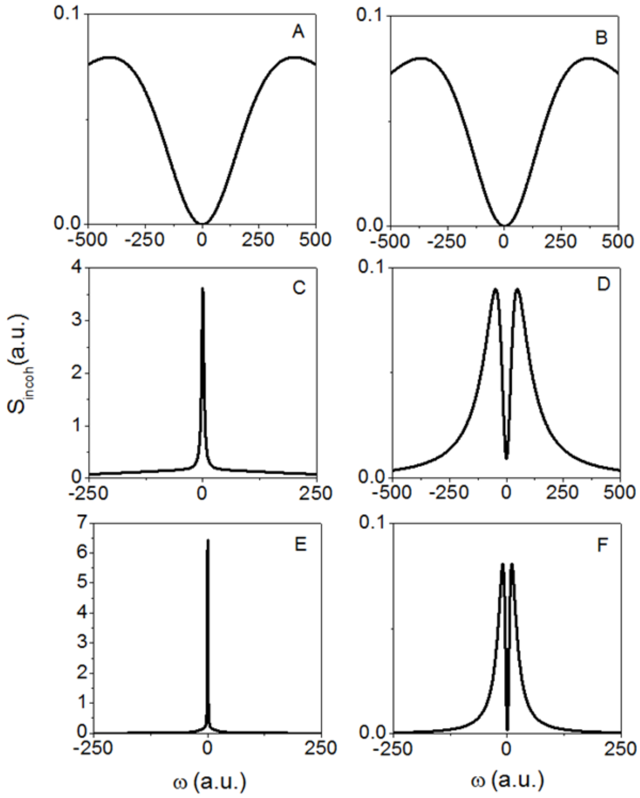

Figure 3. Spectrum of the fluctuations calculated using eq. (57) for marked points of Figure 2 with different values of x_0: *A (x_0=0.25), B (x_0=0.5), C (x_0=1.38), D (x_0=5), E (x_0=8.5), F (x_0=30).*

4. Conclusions

In this paper we demonstrate that in a ring cavity the quantum treatment of optical bistability phenomena occurs for an excitons and biexcitons system. This treatment is done based on the method [14,15] and the cavity is excited via an external coherent field. The excitons, biexcitons and cavity field modes are damped due to the reservoirs interaction of the system. Following the adiabatic elimination of the exciton and biexciton variables, the Fokker-Planck equation for the transmitted field was obtained. In order to obtain this equation, we used the generalized *p* representation that was introduced by Drummond and Gardiner [15]. In this description only the quantum noise is considered, i.e. the fluctuations of the nonlinear interaction between the particles (quasi-particles), the thermal fluctuations that happen due to the reservoir' interaction being neglected. It is worth to mention that quantum noise does not have any classical analogue. Thus, based on the approximation of linearization, the spectrum of transmission and second order correlation function were calculated. The spectrum of transmission describes a hysteresis cycle character, where a narrow spectral line in the end of cycle can be observed. The obtained result is similar to that of optical bistability case for two-level atom system. Finally, we believe that our work provides a good basis for future study, and, in particular, provides some pointers for more detailed experimental and theoretical investigations of optical bistability for a high Q cavity case.

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