An Optimal Landing Problem for a Bessel Process

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Abstract. We consider the one-dimensional controlled diffusion process $\{X(t), t \geq 0\}$ defined by the stochastic differential equation

$$\mathrm{d}X(t) = b_0 \theta u[X(t)]\mathrm{d}t + \frac{(\alpha - 1)}{2X(t)}\mathrm{d}t + \sigma \mathrm{d}B(t), \tag{1}$$

where b_0, θ, α and σ are non-negative constants, the continuous function $u(\cdot)$ is the control variable and $\{B(t), t \geq 0\}$ is a standard Brownian motion. The uncontrolled process $\{X_0(t), t \geq 0\}$ is a Bessel process of dimension α (if σ=1).

Let $T(x)$ be the *first-passage time* defined by

$$T(x) = \inf\{t > 0: X(t) = d \mid X(0) = x > d \geq 0\}. \tag{2}$$

The aim is to find the control $u^*[X(t)]$ that minimizes the expected value of the cost function

$$J(x) = \int_0^{T(x)} \left\{\frac{1}{2} q_0 g(\theta) u^2[X(t)] X^2(t) + \lambda\right\} \mathrm{d}t, \tag{3}$$

where q_0 and λ are positive constants.

This type of problem, in which the optimizer controls a stochastic process until a certain event occurs, is known as a *homing problem*; see [1]-[3]. The above problem can be interpreted as an optimal landing problem, with *d* representing ground level. Because the parameter λ is positive, the optimizer tries to reach *d* as soon as possible, while taking the control costs into account. Therefore, the optimal control $u^*[X(t)]$ should in general be negative. Moreover, θ is a risk parameter. If $\theta < 1$ (respectively, $\theta > 1$) the optimizer is risk-averse (resp., risk-seeking) and does not want to land too rapidly (resp., wants to land rapidly). The case when $\theta = 1$ is the risk-neutral case.

Using dynamic programming, the equation satisfied by the value function

$$F(x) := \inf_{\substack{u[X(t)] \\ 0 \leq t < T(x)}} E[J(x)] \tag{4}$$

is derived. This equation is a non-linear second-order ordinary differential equation. We find that $F(x)$ is actually of the form

$$F(x) = k(x-d)^2, \tag{5}$$

where k is a constant that depends on the various parameters in the model. From the value function, the optimal control is obtained explicitly.

References

[1] M. Lefebvre, LQG homing problems for processes used in financial mathematics, Revue Roumaine de Mathématiques Pures et Appliquées. 63 (2018) 27-37.

[2] P. Whittle, *Optimization over Time*, vol. I. Chichester (UK), Wiley, 1982.

[3] P. Whittle, *Risk-Sensitive Optimal Control,* Chichester (UK), Wiley, 1990.