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# Preservation of radicals by generalizations of derivations

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**Abstract**

By results of Anderson and Slin'ko, derivations preserve the locally nilpotent and nil radicals of algebras over a field of characteristic 0. There is also a well known and elementary result that derivations preserve idempotent ideals. The radical results are extended to some (not all) rings, possible generalizations using generalizations of derivations are examined, the relevance of the result about idempotent ideals is pointed out and some comments about the Jacobson radical are included.

**Keywords:** Algebra, radical, derivation.

## 1 Introduction

Slin'ko [1] showed that if A is an algebra over a field of characteristic 0, then for every algebra derivation $d$ of $A$ we have $d(\mathcal{L}(A) \subseteq \mathcal{L}(A)$ and $d(\mathcal{N}(A) \subseteq \mathcal{N}(A)$, where $\mathcal{L}$, $\mathcal{N}$ are the locally nilpotent and nil radical respectively. Using similar techniques, Anderson [2] had earlier obtained such results for algebras with a chain condition. We generalize these results in two ways, by replacing algebras by rings and replacing derivations by other mappings or sequences of mappings. We are unable to do so for *all* rings and there remain some open questions relating to the action of these generalizations of derivations. Here are the generalizations we shall consider.

A *higher derivation* of a ring $R$ is a sequence $(d_0, d_1, \dots, d_n, \dots)$ of additive endomorphisms of $R$ such that $d_n(ab) = \sum_{i+j=n} d_i(a)d_j(b)$, $a, b \in R$ for every $n$.

If $\alpha$ and $\beta$ are endomorphisms of $R$, then an $(\alpha, \beta)$-*derivation* of $A$ is an additive endomorphism $d$ such that $d(ab) = d(a)\beta(b) + \alpha(a)d(b)$ for all $a, b \in R$, so that an ordinary derivation is an $(id, id)$-derivation.

## 2 Extension to rings

If a ring $R$ is additively torsion-free, we can extend the multiplication of $R$ to the *divisible hull* $D(R)$ of (the additive group of) $R$. Then $D(R)$ becomes an algebra over the field $\mathbb{Q}$ of rational numbers. Moreover, each endomorphism of $R$ extends uniquely to an algebra endomorphism of $D(R)$, and each derivation of $R$ extends uniquely to an algebra derivation. From these results it is fairly straightforward to find analogous extensions for higher derivations and $(\alpha, \beta)$-derivations. We thence get a version of the Anderson-Slin'ko results for these rings.

**Theorem 1.** *If a ring $R$ is additively torsion-free, then $d(\mathcal{L}(R)) \subseteq \mathcal{L}(R)$ and $d(\mathcal{N}(R)) \subseteq \mathcal{N}(R)$ for every derivation $d$ of $R$.*

For torsion rings the situation is unclear. It is routine to show that a radical is preserved on torsion rings if and only if it is preserved on $p$-rings (i.e. rings whose additive groups are $p$-groups), but there are $\mathbb{Z}_p$-algebras for which derivations need not preserve $\mathcal{L}$ and $\mathcal{N}$, by a result of Krempa [3]. For mixed (neither torsion nor torsion-free) rings it is equally unclear.

In the next two sections we shall give results for torsion-free rings. Analogous results for algebras in characteristic 0 hold, but we shall not mention this explicitly. In fact our method of proof is to establish the algebra case first and then proceed to the ring case in imitation of the procedure of this section.

## 3 Higher derivations

If a higher derivation has its zeroth mapping equal to the identity mapping $id$, then all its other mappings are linear combinations of compositions of derivations. This result has been proved many times,

first by Heerema [4] and most recently in a very interesting way by Hazewinkel [5] and is the source for part of the proof of our next result.
**Theorem 2.** *If a ring $R$ is additively torsion-free, then for every higher derivation $(d_0, d_1, \dots, d_n, \dots)$ of $R$ in which $d_0$ is an automorphism we have $d_n(\mathcal{L}(R)) \subseteq \mathcal{L}(R)$ and $d_n(\mathcal{N}(R)) \subseteq \mathcal{N}(R)$ for all $n$.*

For commutative rings without restriction *all* mappings of all higher derivations preserve our two radicals, which coincide with the set of nilpotent elements, but in non-commutative rings higher derivations need not preserve the set of nilpotent elements.

# 4 $(\alpha, \beta)$-derivations

The best result we have here is the following.
**Theorem 3.** *If $\alpha$ is an automorphism of a torsion-free ring $R$, then $d(\mathcal{L}(R)) \subseteq \mathcal{L}(R)$ and $d(\mathcal{N}(R)) \subseteq \mathcal{N}(R)$ for every $(\alpha, \alpha)$-derivation $d$ of $R$.*

It is not known how $(\alpha, \beta)$-derivations treat radicals when $\alpha$ and $\beta$ are unequal automorphisms, but there are examples of non-preservation for non-automorphisms, equal or not.

# 5 The relevance of idempotent ideals

In an arbitrary ring, derivations take idempotent ideals into themselves. There are radical classes which consist entirely of idempotent rings, but these are not our concern. Even $\mathcal{L}, \mathcal{N}$, even the prime radical can take idempotent values. Consider the algebra over a field which has a basis $\{e_t : t \in (0,1)\}$ (here we refer to the real open interval) and multiplication given by $e_t e_u = e_{t+u}$ if $t+u < 1$ and 0 otherwise. This is idempotent and coincides with its prime radical. It is therefore worth stating
**Theorem 4.** *Let $R$ be any ring, $\mathcal{R}$ a radical class. If $\mathcal{R}(R)$ is idempotent, then it is preserved by derivations, $(\alpha, \beta)$-derivations where $\alpha$ and $\beta$ are automorphisms and by higher derivations for which the zeroth mapping is an automorphism.*

# 6 What about the Jacobson radical?

The behaviour of the Jacobson radical with respect to derivations is complicated. In the power series ring (algebra) $\mathbb{Q}[[X]]$ formal differentiation does not preserve it. But Anderson's results show that it is preserved by derivations in algebras (characteristic 0) with DCC on two-sided ideals, and of course the Jacobson radical can be idempotent. We should also mention the Singer-Wermer Theorem [6] as strengthened by others: If $A$ is a complex Banach algebra then for any derivation $d$, the Jacobson radical contains $d(A)$.

Full details of our results will appear elsewhere.

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