The optimal value of one monotone method divisor for PR voting systems

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Abstract: It is described the methodology and determined by simulation the optimal value of the divisor of the Mixed monotone method for decision making by voting with proportional representation – method that exceeds the widely applied Sainte-Laguë one.

Keywords: disproportion, divisor, index, monotone method, optimization, proportional representation, voting systems.

1 Introduction

When taking collective decisions, using voting systems with proportional representation (PR), to minimize the disproportion of deciders' will representation is required – disproportion caused by the character in integers of the number of deciders and that of alternative options. To minimize this disproportion, diverse methods (algorithms, "votes-decision" rules), including the Hamilton (Hare), Sainte-Laguë, d'Hondt and Huntington-Hill ones, are used.

To estimate this disproportion, each method applies an index, which may differ from one method to another. In [2], basing on a comparative multi aspectual analysis, the opportunity of using, in this aim, the Average relative deviation (ARD) index is argued. This index conveys the average relative deviation of the representation in the decision of deciders will from their mean value.

It has been proved [1, 2], that the optimum (minimum) value of ARD index is obtained when using Hamilton method. However, its use can lead, in some cases, to Alabama, of Population or of the New state "paradoxes" [1]. The d'Hondt, Sainte-Laguë and Huntington-Hill [1] methods and the Mixed monotone one, proposed in [3], are immune to these paradoxes.

The Mixed monotone method, at parameter $c = 2$, exceeds the Sainte-Laguë one, in the sense that there are no cases, when the Sainte-Laguë method leads to a better solution in minimizing the ARD index, than the Mixed one, but there are cases, when the Mixed method leads to better solutions, than the Sainte-Laguë one. However, the optimal value of the parameter c, equal to 2, was obtained in [3], basing on some assumptions, which veracity was not strictly proved. In this paper aspects of determining, by simulation, the optimal value of the constant c, are investigated, using, as example of PR voting system, the elections in an elective body on party lists (blocks, coalitions).

2 Mixed monotone method

The problem of minimizing disproportionality in PR systems is formulated as follows [2]. Let: M – number of seats in the elective body; n – number of parties that have reached or exceeded the representation threshold; V – total valid votes cast for the n parties; V_i – total valid votes cast for party i; x_i – number of seats to be allocated to party i; I_d – index of disproportionality (Average relative deviation). Known quantities: M; n; V_i, $i=\overline{1,n}$ and

$$V_1 + V_2 + \ldots + V_n = V. \quad (1)$$

It is required [2] to determine unknowns x_i $(i=\overline{1,n})$ – nonnegative integers, which will assure the I_d minimum value

$$I_d = \frac{\Delta d}{d}100 = \sum_{i=1}^{n}\left|v_i - m_i\right| = \frac{100}{V}\sum_{i=1}^{n}\left|V_i - Qx_i\right| \to \min, \quad (2)$$

where $\Delta d = \frac{1}{V}\sum_{i=1}^{n} V_i\left|d_i - d\right|$, and $Q = V/M = 1/d$ is the simple quota, named the Hare one, too [1], $v_i = 100\cdot V_i/V$ (%) and $m_i = 100\cdot x_i/M$ (%), in compliance with the restriction:

$$x_1 + x_2 + \ldots + x_n = M. \quad (3)$$

The Mixed monotone method for solving problem (2)-(3) is the following [3]:

1. Calculate the quantities

$$a_i := \lceil V_i/Q \rceil,\ i = \overline{1,n}, \quad (4)$$

where $\lceil z \rceil$ signifies the integer of number z. Afterwards, determine the number ΔM of still undistributed seats

$$\Delta M = M - (a_1 + a_2 + \ldots + a_n). \quad (5)$$

If $\Delta M = 0$, then the distribution has been completed and is proportional.

2. Otherwise, the ΔM seats, remaining undistributed after the first step, to assign, by one, to each of the first ΔM parties with the larger ratio $V_i/(ca_i + 1)$.

3 Aspects of RP systems simulation methodology

From the initial data of the problem (2)-(3), subject to simulation are, basically, only quantities V_i, $i = \overline{1,n}$, the sum of which, for each ballot, must be equal to V. Given this constraint, to generate values for V_i, $i = \overline{1,n}$, the following procedure is proposed:

1. Randomly generate n numbers N_i, $i = \overline{1,n}$ in the interval (0; 1).
2. To determine $w_i = \left\lfloor VN_i / \sum_{j=1}^{n} N_j \right\rfloor i = \overline{1,n}$.
3. If, considering $V_i = w_i$, $i = \overline{1,n}$, the condition (1) occurs, then quantities w_i, $i = \overline{1,n}$ to order in decreasing, thus obtaining the expected values of quantities V_i, $i = \overline{1,n}$ (stop).
4. To determine $\Delta W_i = \left\lfloor VN_i / \sum_{j=1}^{n} N_j - w_i \right\rfloor i = \overline{1,n}$ and $\Delta W = \sum_{i=1}^{n} \Delta W_i$.
5. From the n values ΔW_i, to select ΔW highest values ΔW_i and for each of them to determine $W_i = w_i + 1$, and for the other $n - \Delta W$ cases to set $W_i = w_i$.
6. Quantities W_i, $i = \overline{1,n}$ to order in decreasing, thus obtaining the expected values of quantities V_i, $i = \overline{1,n}$. Stop.

For done sample size, the software application SIMRP calculates the average value $\bar{I}_d$ (c) of index $I_d(c)$. The uniform distribution and, separate, the normal one for numbers N_i, $i = \overline{1, n}$ in the interval (0, 1) are investigated.

4 Results of calculations for obtaining the value of *c*

The initial data, used for the application SIMRP, are: M = 20, 50, 100, 1000; n = 2, 4, 6, 8, 10; $V = 10^8$; sample size 20000. For each pair of values {M; n}, the quasi optimal value of parameter c, that assures the lowest value of the index $\bar{I}_d$, is calculated five times (five probes, $k = \overline{1,5}$).

Some results of the calculations of parameter c quasi optimal value, for $5 \cdot 5 \cdot 4 \cdot 2 \cdot 10^4 = 10^6$ ballots, are presented in Table 1 (normal distribution).

Table 1 Results of parameter *c* calculation (normal distribution)

M	n	Probes for determining c (c_k, $k = \overline{1,5}$)					c_{med}	δ,%
		1	2	3	4	5		
20	2	2	2	2	2	2	2	0
	4	2,04	2,03	2,03	1,99	2	2,018	0,9
	6	2,02	2,05	2,03	2,06	2,02	2,036	1,8
	8	1,99	2,01	2,05	2,04	2,02	2,022	1,1
	10	2,06	2,02	2,01	2,04	2,05	2,036	1,8
50	2	2	2	2	2	2	2	0
	4	1,98	1,95	1,99	1,96	1,95	1,966	-1,7
	6	1,99	2,02	1,97	1,97	1,98	1,986	-0,7
	8	1,98	2,01	2,02	2	2	2,002	0,1
	10	1,99	2	2,04	2,03	2,03	2,018	0,9
100	2	2	2	2	2	2	2	0
	4	2,04	2,01	2	2,04	2,03	2,024	1,2
	6	1,98	1,99	2,01	2,01	1,99	1,996	-0,2
	8	2	2	2,02	2	2,01	2,006	0,3
	10	1,99	1,99	2,02	2,04	2,01	2,01	0,5
1000	2	2	2	2	2	2	2	0
	4	2	2	2,01	1,97	1,99	1,994	-0,3
	6	1,98	1,98	2,03	2,02	1,99	2	0
	8	1,96	1,96	2,01	2	2,01	1,988	-0,6
	10	1,98	1,98	1,99	2	2	1,99	-0,5
Average							2,0046	0,23

From Table 1 one can see that at n = 2 takes place c_k = 2, $k = \overline{1,5}$. Such a situation results from relation [4]

$$c = n/\Delta M. \quad (6)$$

Indeed, at n = 2, taking into account that $\Delta M \in [1; n-1]$ (the case ΔM = 0 ensure the proportional distribution), one has ΔM = 1 and replacing in (6), we obtain c = 2.

Thus, the average size c_{med} of parameter c is 2,0046, at a relative deviation δ of c_{med} from 2, equal to 0,23%. Here, takes place:

$$c_{med} = \sum_{k=1}^{5} c_k / 5; \ \delta = (c_{med} / 2 - 1)100\%.$$

For the uniform distribution, similar calculations lead to: c_{med} = 2,005; δ = 0,25% – values falling into error of simulations.

5 Conclusion

The optimum value of parameter c, for the Mixed monotone method, proposed in [3], is equal to 2, both at the normal distribution and at the uniform one of the number of votes V_i, $i = \overline{1, n}$.

References

[1] Gallagher, M.; Mitchell, P. *The Politics of Electoral Systems*. London: Oxford University Press, 2008. 672 p.

[2] Bolun, I. *Algorithmization of optimal allocation of seats in PR systems*. In: Economica. 2011, vol. 77, nr.3. Chişinău: Editura ASEM, p. 137-152.

[3] Bolun, I. *"Votes-decision" monotone method in PR systems.* In: Economica, nr.4(78)/2011. Chişinău: Editura ASEM, p. 108-117.

[4] Bolun I. Disproportionality of some "votes-decision" rules in PR systems. In: ASEM: 20 years of ascension. - Chişinău: Editura ASEM, 2011 (Romanian).

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