The geometry of fractional tangent bundle and applications

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Abstract. The authors present the use of the revised fractional Riemann-Liouville derivative in the fractional tangent bundle of order k, $\overset{\alpha k}{T}M$, of a differential manifold M and the behavior of some objects under a change of local map. Among the geometrical structures defined on $\overset{\alpha k}{T}M$ we consider the fractional connections and the fractional Euler-Lagrange equations associated to a function defined on $\overset{\alpha k}{T}M$. Some examples are given.

M.S.C. 2000: 26A33, 53C05, 70H03.
Key words: Revised Riemann-Liouville fractional calculus on manifolds, fractional tangent bundle of higher order, fractional geometrical object, fractional Euler-Lagrange equation.

1 Introduction

It is known that the operators of fractional integration and derivation have geometrical and physical interpretations (see [7]) and they have been used in the modelation of problems in various domains. For example, in [6], the use of the revised Riemann-Liouville integral and derivative is shown.

In this paper, the fractional differential calculus on a differential manifold and the principal geometrical structures on the fractional tangent bundle of order k are considered. We also define the Euler-Lagrange equations associated to a function on that bundle.

Section 2 contains the definitions of the fractional operators on $\mathbb{R}$ and some of their properties which are used in the sequel.

In Section 3, we describe the principal elements of fractional differential calculus on a manifold.

The fractional tangent bundle of order k, $\overset{\alpha k}{T}M$ ($\alpha > 0$) and some structures having a geometric character are defined in Section 4.

BSG Proceedings 16. The Int. Conf. of Diff. Geom. and Dynamical Systems (DGDS-2008) and The V-th Int. Colloq. of Mathematics in Engineering and Numerical Physics (MENP-5) - math. sections, August 29 - September 2, 2008, Mangalia, Romania, pp. 1-11.

In Section 5, the fractional Euler-Lagrange equations are established by using the notions of classical extremal value and fractional extremal value of an action. We also give two examples which prove that certain equations do not admit classical Lagrangians but they can be considered as fractional Euler-Lagrange equations.

2 Brief overview of fractional differential calculus

There are many books dealing with the fractional differential calculus and various definitions of the fractional integration and derivation. For the purpose of this note the revised Riemann-Liouville operators will be used (see [**?**]).

Let $\Gamma(\alpha)$ be the gamma function of the parameter α. α is not necessarily an integer and $\Gamma(1+\alpha)=\alpha!$ if α is natural. For a function $f: I \subset \mathbb{R} \to \mathbb{R}$ with $0 \in I$ the *fractional derivative of order* α is defined by the expressions:

$$D_t^\alpha f(t) := \frac{1}{\Gamma(-\alpha)} \int_0^t \frac{f(s)-f(0)}{(t-s)^{1+\alpha}} ds \quad \text{for } \alpha < 0; \tag{2.1}$$

$$D_t^\alpha f(t) := \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_0^t \frac{f(s)-f(0)}{(t-s)^{\alpha-m+1}} ds \quad \text{for } \alpha \geq 0, \tag{2.2}$$

where m is the first integer greater than or equal to α. The relation (2.1) defines a fractional integral and (2.2) gives a fractional derivative. The last one has many interesting properties:

$$D_t^\alpha t^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} t^{\gamma-\alpha}, \quad \gamma > -1,\ 0 \leq \alpha < 1; \tag{2.3}$$

$$D_t^n D_t^\alpha f(t) = D_t^{n+\alpha} f(t), \quad n \in \mathbb{N}; \tag{2.4}$$

$$D_t^\alpha D_t^{-\alpha} f(t) = f(t), \qquad D_t^{-\alpha} D_t^\alpha f(t) \neq f(t); \tag{2.5}$$

$$D_t^\alpha (fg)(t) = \sum_{i=0}^\infty \left(\begin{array}{c} \alpha \\ i \end{array} \right) D_t^{\alpha-i} f(t) D_t^i g(t), \tag{2.6}$$

where $D_t^i = \frac{d}{dt} \circ \frac{d}{dt} \circ \cdots \circ \frac{d}{dt}$ (i times).

If f is an analytical function then

$$f(t) = \sum_{i=0}^\infty E_\alpha(t) D_t^{\alpha i} f(t), \tag{2.7}$$

where $E_\alpha(t)$ is the *Mittag-Leffler function*,

$$E_\alpha(t) := \sum_{j=0}^\infty \frac{t^{\alpha j}}{\Gamma(1+\alpha j)}. \tag{2.8}$$

3 Fractional differential calculus on manifolds

Let M be an n-dimensional differential manifold, (U, x^i) a local coordinate system on M and $U_0 = \{x \in U \mid 0 \leq x^i \leq b^i,\ i = 1, 2, \ldots, n\}$.

For a function $f : U_0 \to \mathrm{I\!R}$ we define *the fractional derivative with respect to* x^i:

$$D^{\alpha}_{x^i} f(x) := \tag{3.1}$$

$$= \frac{1}{\Gamma(m-\alpha)} D^m_{x^i} \int_0^{x^i} \frac{f(x^1, \ldots, x^{i-1}, s, x^{i+1}, \ldots, x^n) - f(x^1, \ldots, x^{i-1}, 0, x^{i+1}, \ldots, x^n)}{(x^i - s)^{\alpha - m + 1}} ds,$$

where $D^m_{x^i} = \dfrac{\partial}{\partial x^i} \circ \dfrac{\partial}{\partial x^i} \circ \cdots \circ \dfrac{\partial}{\partial x^i}$ (m times), i is fixed, $\alpha \geq 0$.

In particular we have, for $\alpha \in (0,1)$, $\gamma > -1$,

$$D^{\alpha}_{x^i} (x^i)^{\gamma} = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)}; \qquad D^{\alpha}_{x^i} \frac{x^j}{\Gamma(1+\alpha)} = \delta^j_i.$$

Let $D_{x^i} := \dfrac{\partial}{\partial x^i}$, $i = 1, 2, \ldots, n$, denote the local base of the module of vector fields $\mathcal{X}_U(M)$ and let dx^i, $i = 1, 2, \ldots, n$, be the local base of the 1-forms $\mathcal{D}^1_U(M)$. Using (3.1) and the previous relations, we get

Proposition 3.1. *Let* (U, x^i), $\left(\overline{U}, \overline{x}^i\right)$ *be two local coordinate systems such that* $U \cap \overline{U} \neq \varnothing$ *and*

$$\overline{x}^i = \overline{x}^i(x^1, x^2, \ldots, x^n),\ \text{rang}\left(\frac{\partial \overline{x}^i}{\partial x^i}\right) = n, \tag{3.2}$$

the corresponding coordinate transformation. The following relations hold:

$$\begin{aligned} & d\overline{x}^i = J^i_j(\overline{x}, x) dx^j,\ \textit{where } J^i_j(\overline{x}, \overline{x}) = D_{x^j} \overline{x}^i; \\ & D^{\alpha}_{x^j} = \overset{\alpha}{J}{}^i_j(\overline{x}, x) D^{\alpha}_{\overline{x}^i},\ \textit{where } \overset{\alpha}{J}{}^i_j(\overline{x}, x) = D^{\alpha}_{x^j} \overline{x}^i; \\ & J^i_j(\overline{x}, x) \overset{\alpha}{J}{}^j_h(\overline{x}, x) = \delta^i_h. \end{aligned} \tag{3.3}$$

A *fractional vector field* on $U \subset M$ is an object of the form

$$\overset{\alpha}{X} = \overset{\alpha}{X}{}^i D^{\alpha}_{x^i},\ \text{where } \overset{\alpha}{X}{}^i \in \mathcal{F}_U(M),\ i = 1, 2, \ldots, n.$$

We denote by $\overset{\alpha}{\mathcal{X}}_U$ the module of the fractional vector fields on U. $\overset{\alpha}{\mathcal{X}}_U$ is generated by the operators $D^{\alpha}_{x^i}$, $i = 1, 2, \ldots, n$. If $c : x = x(t)$, $t \in I$ is a parametrized curve in U then *the fractional tangent field of* c *is* given by

$$\overset{\alpha}{x}(t) = \frac{1}{\Gamma(1+\alpha)} D^{\alpha}_t x^i(t) D^{\alpha}_{x^i}. \tag{3.4}$$

Remark 3.1. Under a change of local coordinate systems (3.2) we have

$$\overline{\overset{\alpha}{X}}{}^i = \overset{\alpha}{J}{}^i_j(\overline{x}, x) \overset{\alpha}{X}{}^j. \tag{3.5}$$

Remark 3.2. To each fractional vector field $\overset{\alpha}{X}$ we may associate a *fractional differential equation of order* α given by

$$\overset{\alpha}{x}(t) = \overset{\alpha}{X}(x(t)). \tag{3.6}$$

The equation (3.6) with an initial condition has a solution.

There are several fractional differential equations (systems) which are studied in literature because of their chaotic behaviour. Among the famous ones we mention: the fractional Lorenz system (1963), the fractional Rössler system (1976), Rabinovich - Fabrikant system (1979), Chua's system (1983) and so on.

Let us consider the real function U given by

$$x^i_\alpha := \frac{1}{\Gamma(1+\alpha)}(x^i)^\alpha \in \mathcal{F}_U, \quad i = 1, 2, \ldots, n.$$

According to [3] we can define *the fractional exterior derivative* $d_\alpha : \mathcal{F}_U \to \mathcal{D}^1_U$ by

$$d_\alpha f := D^\alpha_{x^i} f dx^i_\alpha, \quad f \in \mathcal{F}_U. \tag{3.7}$$

Remark 3.3. With respect to the change of the local coordinate system (3.2) we have

$$d_\alpha \overline{x}^i = \overset{\alpha}{J}{}^i_j(\overline{x}, x) d_\alpha x^i. \tag{3.8}$$

In order to extend the operator d_α on forms we determine the action of $D^\alpha_{x^i}$ on $\mathcal{D}^1_U$. If $\omega = \omega_j dx^j \in \mathcal{D}^1_U$ then

$$D^\alpha_{x^i}(\omega) = D^\alpha_{x^i}(\omega_j dx^j) = \sum_{k=0}^{\infty} \begin{pmatrix} \alpha \\ k \end{pmatrix} D^{\alpha-k}_{x^i}(\omega_j) D^k_{x^i}(dx^j). \tag{3.9}$$

Since $D^k_{x^i}(dx^j) = d(D^k_{x^i} x^j) = 0$ for $k \geq 1$, it results

$$D^\alpha_{x^i}(\omega) = D^\alpha_{x^i}(\omega_j) dx^j. \tag{3.10}$$

We now define *the operator* $d_\alpha : \mathcal{D}^1_U \to \mathcal{D}^2_U$ by

$$d_\alpha \omega := dx^i_\alpha \wedge D^\alpha_{x^i}(\omega), \quad \omega \in \mathcal{D}^1_U. \tag{3.11}$$

¿From (3.10) and (3.11) we obtain

$$d_\alpha \omega = \frac{\alpha (x^i)^{\alpha-1}}{\Gamma(1+\alpha)} D^\alpha_{x^i}(\omega_j) dx^i \wedge dx^j \in \mathcal{D}^2_U. \tag{3.12}$$

Remark 3.4. Using (3.12) one verifies that

$$d_\alpha \circ d_\alpha = 0. \tag{3.13}$$

In this way the classical differential calculus on manifolds may be extended to a fractional differential calculus taking into account the operators $D^\alpha_{x^i}$, d_α and the following property of a fractional vector field $\overset{\alpha}{X} = \overset{\alpha}{X}{}^i D^\alpha_{x^i} \in \overset{\alpha}{\mathcal{X}}_U$:

$$\overset{\alpha}{X}(fg)(x) = \sum_{k=0}^{\infty} \begin{pmatrix} \alpha \\ k \end{pmatrix} \overset{\alpha}{X^i}(x) D_{x^i}^{\alpha-k} f(x) D_{x^i}^{k} g(x), \quad f, g \in \mathcal{F}_U. \tag{3.14}$$

Given $\overset{\alpha}{X}, \overset{\alpha}{Y} \in \overset{\alpha}{\mathcal{X}}_U$ we can still consider a *covariant derivative* by the formula

$$\overset{\alpha}{\nabla}_{\overset{\alpha}{X}} \overset{\alpha}{Y} = \overset{\alpha}{X^i} \left(D_{x^i}^{\alpha} \overset{\alpha}{Y}{}^j + \overset{\alpha}{\Gamma}{}^j_{ik} \overset{\alpha}{Y}{}^k \right) D_{x^j}^{\alpha}, \tag{3.15}$$

where $\left(\overset{\alpha}{\Gamma}{}^j_{ik}\right)$ are the functions defining *the coefficients of a fractional linear connection on M*. They are determined by the relations

$$\overset{\alpha}{\nabla}_{D_i^\alpha} D_k^\alpha = \overset{\alpha}{\Gamma}{}^j_{ik} D_j^\alpha, \tag{3.16}$$

where $D_i^\alpha := D_{x^i}^\alpha$, $i, j, k = 1, 2, \ldots, n$.

With respect to a change of local coordinates on M, the coefficients (Γ^j_{ik}) change according to the classical law by using $\overset{\alpha}{J}{}^i_j(\overline{x}, x)$ instead of $J^i_j(\overline{x}, x)$.

4 The fractional tangent bundle $\overset{\alpha k}{T} M$. Geometrical structure

Let $\alpha \in (0, 1)$ be fixed and two parametrized curves $c_1, c_2 : I \to U \subset M$, $0 \in I$, with $c_1(0) = c_2(0) = x_0 \in U$, x_0 fixed.

We say that c_1 and c_2 *have a fractional contact of order $k \in \mathbb{N}^*$ in x_0* if for any $f \in \mathcal{F}_U$

$$D_t^{\alpha a}(f \circ c_1)\Big|_{t=0} = D_t^{\alpha a}(f \circ c_2)\Big|_{t=0}, \quad a = 1, 2, \ldots, k \tag{4.1}$$

holds.

The conditions (4.1) define an equivalence relation on the parametrized curves around x_0.

An equivalence class $[c]^{\alpha k}_{x_0}$ is called *a fractional k-tangent vector to M in x_0*. The set of these classes is *the fractional k-tangent space of M in x_0* and it will be denoted by $\overset{\alpha k}{T}_{x_0} M$.

By considering $\overset{\alpha k}{T} M := \bigcup_{x_0 \in M} \overset{\alpha k}{T}_{x_0} M$ and the map

$$\overset{\alpha k}{\pi}_0 : [c]^{\alpha k}_{x_0} \in \overset{\alpha k}{T} M \mapsto x_0 \in M$$

we obtain a bundle

$$\left(\overset{\alpha k}{T} M, \overset{\alpha k}{\pi}_0, M \right). \tag{4.2}$$

There is a differential structure on $\overset{\alpha k}{T}$ (see [2]) and this bundle is called *the fractional tangent bundle of order k of M*.

We make two specifications:

1. If the curve c has the local representation $c: x^i = x^i(t)$, $t \in I$, $i = 1, 2, \ldots, n$, in (U, x^i), then the class $[c]_{x_0}^{\alpha k}$ is given by

$$x^i(t) = x^i(0) + \sum_{a=1}^{k} \frac{t^{\alpha a}}{\Gamma(1+\alpha a)} D_t^{\alpha a} x^i(t)\Big|_{t=0}, \ i = 1, 2, \ldots, n, \tag{4.3}$$

where $t \in (-\varepsilon, \varepsilon) \subset i$.

2. Using the notations

$$x^i = x^i(0), \quad \overset{a\alpha_i}{x} = \frac{1}{\Gamma(1+\alpha a)} D_t^{\alpha a} x^i(t)\Big|_{t=0}, \ i = 1, 2, \ldots, n; \ a = 1, 2, \ldots, k, \tag{4.4}$$

we have *the induced local coordinate system on* $\overset{\alpha k}{T} M$,

$$\left(\left(\overset{\alpha k}{\pi_0}\right)^{-1}(U); x^i, \overset{\alpha}{x^i}, \overset{2\alpha}{x^i}, \ldots, \overset{k\alpha}{x^i}\right). \tag{4.5}$$

We can establish

Proposition 4.1. *With respect to a change of local coordinates* (3.2) *on* M *the following relations hold:*

$$\frac{\Gamma((a-1)\alpha)}{\Gamma(\alpha)} \overset{a\alpha}{\overline{x}^i} = \Gamma(1+\alpha) \overset{\alpha}{J_j^i}\left(\overset{(a-1)\alpha}{\overline{x}}, x\right) \overset{\alpha}{x^i} + \tag{4.6}$$

$$+ \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} \sum_{b=1}^{a-1} \overset{\alpha}{J}{}_j^i \left(\overset{(a-1)\alpha}{\overline{x}}, \overset{b\alpha}{x^j}\right) \overset{(b+1)\alpha}{x} + \frac{\Gamma((a-1)\alpha)}{\Gamma(\alpha)} \overset{a\alpha_i}{x},$$

$a = 1, 2, \ldots, k$*;* $b = 2, 3, \ldots, k$*,* $b \leq a$*, where*

$$\begin{aligned} \overset{\alpha}{J_j^i}\left(\overset{(a-1)\alpha}{\overline{x}}, \overset{b\alpha}{x}\right) &= D^{\alpha}_{\overset{b\alpha}{x^j}} \overset{(a-1)\alpha}{\overline{x}^i}, \\ \overset{\alpha}{J_j^i}\left(\overset{(a-1)\alpha}{\overline{x}}, x\right) &= D^{\alpha}_{x^j} \overset{(a-1)\alpha}{\overline{x}^i}, \ i, j = 1, 2, \ldots, k. \end{aligned} \tag{4.7}$$

Remark 4.1. The fractional tangent bundle of order k is a generalization of the tangent bundle of order k. If $\alpha_p \in (0, 1)$ is a sequence such that $\lim_{p\to\infty} \alpha_p = 1$, then $\lim_{p\to\infty} \overset{\alpha_p k}{T} M = \overset{k}{T} M$ or $T^k M$.

Let us define other geometrical structures and objects on the fractional tangent bundle $\overset{\alpha k}{T} M$.

1. Let $\pi_{\alpha h}^{\alpha k} : \left(x, \overset{\alpha}{x}, \ldots, \overset{k\alpha}{x}\right) \in \overset{\alpha k}{T} M \mapsto \left(x, \overset{\alpha}{x}, \ldots, \overset{h\alpha}{x}\right) \in \overset{\alpha h}{T} M$, $h < k$, be the natural projection and

$$d_\alpha \pi_{\alpha h}^{\alpha k} : \overset{\alpha}{\mathcal{X}}\left(\overset{\alpha k}{T} M\right) \longrightarrow \overset{\alpha}{\mathcal{X}}\left(\overset{\alpha h}{T} M\right),$$

the operator given by

$$d_\alpha \pi^{\alpha k}_{\alpha h} := \Gamma(1+\alpha)\left[dx^i_\alpha D^\alpha_{x^i} + \sum_{a=1}^{h} dx^i_\alpha D^\alpha_{\overset{a\alpha}{x}{}^i}\right], \quad h < k, \tag{4.8}$$

where the module $\overset{\alpha}{\mathcal{X}}\left(\overset{\alpha k}{T}M\right)$ of fractional vector fields on $\overset{\alpha k}{T}M$ is generated by $D^\alpha_{x^i}$, $D^\alpha_{\overset{a\alpha}{x^i}}$, $i = 1, 2, \ldots, n$; $a = 1, 2, \ldots, k$.

If we consider $V^{\alpha k}_{\alpha h} := \ker d_\alpha \pi^{\alpha k}_{\alpha h}$, $h = 0, 1, \ldots, k-1$, then

$$\begin{aligned} & V^{\alpha k}_{\alpha(k-1)} \subset V^{\alpha k}_{\alpha(k-2)} \subset \cdots \subset V^{\alpha k}_{\alpha 1} \subset V^{\alpha k}_0; \\ & d_\alpha \pi^{\alpha k}_{\alpha h}(D^\alpha_{x^i}) = D^\alpha_{x^i};\ d_\alpha \pi^{\alpha k}_{\alpha h}\left(D^\alpha_{\overset{b\alpha}{x^i}}\right) = D^\alpha_{\overset{b\alpha}{x^i}},\ b = 1, 2, \ldots, k. \end{aligned} \tag{4.9}$$

By Proposition 4.1, $V^{\alpha k}_{\alpha h}$ has a geometrical character.

2. The following fractional vector fields:

$$\begin{aligned} & \overset{\alpha}{\Gamma} = \Gamma(1+\alpha)\overset{\alpha}{x^i}\, D^\alpha_{\overset{k\alpha}{x^i}}, \\ & \overset{2\alpha}{\Gamma} = \Gamma(1+\alpha)\overset{\alpha}{x^i}\, D^\alpha_{\overset{(k-1)\alpha}{x^i}} + \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} D^\alpha_{\overset{k\alpha}{x^i}}, \\ & \ldots \\ & \overset{k\alpha}{\Gamma} = \Gamma(1+\alpha)\overset{\alpha}{x^i}\, D^\alpha_{\overset{\alpha}{x^i}} + \frac{\Gamma(2\alpha)}{\Gamma(\alpha)}\overset{2\alpha}{x^i}\, D^\alpha_{\overset{2\alpha}{x^i}} + \cdots + \frac{\Gamma((k-1)\alpha)}{\Gamma(\alpha)}\overset{k\alpha}{x^i}\, D^\alpha_{\overset{k\alpha}{x^i}}, \end{aligned} \tag{4.10}$$

are called *the Liouville fractional vector fields.*

The operator $\overset{\alpha k}{\Lambda} : \overset{\alpha}{\mathcal{X}}\left(\overset{\alpha k}{T}M\right) \to \overset{\alpha}{\mathcal{X}}\left(\overset{\alpha k}{T}M\right)$ with the properties

$$\overset{\alpha k}{\Lambda}(D^\alpha_{x^i}) = D^\alpha_{\overset{\alpha}{x^i}};\ \overset{\alpha k}{\Lambda}\left(D^\alpha_{\overset{a\alpha}{x^i}}\right) = D^\alpha_{\overset{(a+1)\alpha}{x^i}},\ a = 1, 2, \ldots, k-1;\ \overset{\alpha k}{\Lambda}\left(D^\alpha_{\overset{k\alpha}{x^i}}\right) = 0 \tag{4.11}$$

determine a *k-fractional tangent structure on* $\overset{\alpha k}{T}M$*.*

Using (4.10), (4.11) we obtain

Proposition 4.2. *The k-fractional tangent structure has the following properties:*

a) $\overset{\alpha k}{\Lambda}$ *has a geometrical character;*

b) rang $\overset{\alpha k}{\Lambda} = kn$, $\overset{\alpha k}{\Lambda} \circ \overset{\alpha k}{\Lambda} \circ \cdots \circ \overset{\alpha k}{\Lambda} = 0$;

c) $\overset{\alpha k}{\Lambda}\left(\overset{k\alpha}{\Gamma}\right) = \overset{(k-1)\alpha}{\Gamma}, \ldots, \overset{\alpha k}{\Lambda}\left(\overset{2\alpha}{\Gamma}\right) = \overset{\alpha}{\Gamma},\ \overset{\alpha k}{\Lambda}\left(\overset{\alpha}{\Gamma}\right) = 0.$

3. A fractional vector field $\overset{\alpha k}{S} \in \overset{\alpha}{\mathcal{X}}\left(\overset{\alpha k}{T}M\right)$ is called a *k-fractional spray* if $\overset{\alpha k}{\Lambda}\left(\overset{\alpha k}{S}\right) = \overset{k\alpha}{\Gamma}$. From the last relation we obtain the expresion of the k-fractional spray:

$$\overset{\alpha k}{S} = \Gamma(1+\alpha)\overset{\alpha}{x^i}\, D^\alpha_{x^i} + \sum_{b=2}^{k-1}\frac{\Gamma(b\alpha)}{\Gamma(\alpha)}\overset{b\alpha}{x^i}\, D^\alpha_{\overset{(b-1)\alpha}{x^i}} - \frac{\Gamma(k\alpha)}{\Gamma(\alpha)} G^i\left(x, \overset{\alpha}{x}, \ldots, \overset{k\alpha}{x}\right) D^\alpha_{\overset{k\alpha}{x^i}}. \tag{4.12}$$

There results

Proposition 4.3 *The k-fractional spray* $\overset{\alpha k}{S}$ *uniquely defines the fractional differential equations given by:*

$$\frac{1}{\Gamma(1+k\alpha)}D_t^{(k+1)\alpha}x^i(t)+G^i\left(x,D_t^\alpha x,\ldots,\Gamma(1+(k-1)\alpha)D_t^{k\alpha}x\right)=0. \tag{4.13}$$

5 Fractional Euler-Lagrange equations on $\overset{\alpha k}{T}M$

Let $c:t\in[0,1]\mapsto x(t)\in U\subset M$ be a parametrized curve on M and

$$\overset{\alpha k}{c}:t\in[0,1]\mapsto\left(x(t),\overset{a\alpha}{x}(t)\right)\in\left(\overset{dk}{\pi}{}_0\right)^{-1}(U)\subset\overset{\alpha k}{T}M,\ a=1,2,\ldots,k$$

the extension of c to $\overset{\alpha k}{T}M$.

Let $L:\overset{\alpha k}{T}M\to\mathbb{R}$ be a *fractional Lagrange function. The action of L along the curve* $\overset{\alpha k}{c}$ is

$$\mathcal{A}\left(\overset{\alpha k}{c}\right)=\int_0^1 L\left(x(t),\overset{a\alpha}{x}(t)\right)dt. \tag{5.1}$$

Consider a family of curves

$$c_\varepsilon:t\in[0,1]\mapsto x(t,\varepsilon)\in M$$

where the absolute value of $\varepsilon\geq 0$ is sufficiently small such that $\operatorname{Im}c_\varepsilon\subset U\subset M$, $c_0(t)=c(t)$, $D_\varepsilon^\alpha c_\varepsilon\Big|_{\varepsilon=0}=0$.

The action of L on the curve $\overset{\alpha k}{c_\varepsilon}$ is given by

$$\mathcal{A}\left(\overset{\alpha k}{c_\varepsilon}\right)=\int_0^1 L\left(x(t,\varepsilon),\overset{a\alpha}{x}(t,\varepsilon)\right)dt, \tag{5.2}$$

where $\overset{a\alpha}{x}(t,\varepsilon)=\dfrac{1}{\Gamma(1+a\alpha)}D_t^{\alpha a}x(t,\varepsilon),\quad a=1,2,\ldots,k.$

The action (5.2) has a fractional extremum if

$$D_\varepsilon^\alpha\left[\mathcal{A}\left(\overset{\alpha k}{c_\varepsilon}\right)\right]\Big|_{\varepsilon=0}=0. \tag{5.3}$$

At the same time, the action (5.2) has an extremal value if

$$D_\varepsilon\left[\mathcal{A}\left(\overset{\alpha k}{c_\varepsilon}\right)\right]\Big|_{\varepsilon=0},\ \text{(see [1])}. \tag{5.4}$$

On the basis of the conditions (5.3) and (5.4) we have the following

Proposition 5.1. a) *A necessary condition for the action* (5.2) *to reach a fractional extremal value is that* $c:x=x(t)$ *satisfy the fractional Euler-Lagrange equations:*

$$D^{\alpha}_{x^i} L + \sum_{a=1}^{k} (-1)^a d^{\alpha a}_t \left(D^{\alpha}_{\overset{a\alpha}{x^i}} L \right) = 0, \quad i = 1, 2, \ldots, n, \tag{5.5}$$

where

$$d^{\alpha a}_t = \sum_{b=1}^{a} \overset{b\alpha}{x^i} D^{\alpha}_{\overset{(b-1)\alpha}{x^i}}; \quad D^{\alpha}_{\overset{\circ}{x}^i} \equiv D^{\alpha}_{x^i}. \tag{5.6}$$

b) *A necessary condition for the action* (5.2) *to reach an extremal value is that* $c : x = x(t)$ *satisfy the fractional Euler-Lagrange equations:*

$$D_{x^i} L + \sum_{a=1}^{k} (-1)^a d^a_t \left(D^{\alpha}_{\overset{a\alpha}{x^i}} L \right) = 0, \quad i = 1, 2, \ldots, n, \tag{5.7}$$

where

$$d^a_t = \sum_{b=1}^{a} \overset{b\alpha}{x^i} D_{\overset{(b-1)\alpha}{x^i}}; \; D_{\overset{\circ}{x}^i} \equiv D_{x^i}. \tag{5.8}$$

Examples and applications. 1. Let $L : \overset{2\alpha}{T} \mathbb{R} \to \mathbb{R}$ be the Lagrange function given by

$$L \left(x, \overset{\alpha}{x}, \overset{2\alpha}{x} \right) = -\psi_1(x) - \frac{1}{2} a_1 \Gamma(1 + 2\alpha) \left(\overset{\alpha}{x} \right)^2 - \frac{1}{2} \Gamma(1 + 4\alpha) \left(\overset{2\alpha}{x} \right)^2, \tag{5.9}$$

where $\dfrac{d\psi_1}{dx} =: \varphi_1(x)$. The fractional Euler-Lagrange equation associated to L is

$$\Gamma(1 + 4\alpha) \overset{4\alpha}{x} + a_1 \Gamma(1 + 2\alpha) \overset{2\alpha}{x} + \varphi_1(x) = 0. \tag{5.10}$$

If $\alpha = \dfrac{1}{2}$, the equation (5.10) is

$$\ddot{x}(t) + a_1 \dot{x}(t) + \varphi_1(x(t) = 0. \tag{5.11}$$

It does not admit a classical Lagrangean but it comes from a fractional Lagrangean.

2. Let $L : \overset{3\alpha}{T} \mathbb{R} \to \mathbb{R}$ be the Lagrange function given by

$$L \left(x, \overset{\alpha}{x}, \overset{2\alpha}{x}, \overset{3\alpha}{x} \right) = -\psi_2(x) + \frac{1}{2} a_1 \Gamma(1 + 2\alpha) \left(\overset{\alpha}{x} \right)^2 -$$

$$-\frac{1}{2} a_2 \Gamma(1 + 4\alpha) \left(\overset{2\alpha}{x} \right)^2 + \frac{1}{2} \Gamma(1 + 6\alpha) \left(\overset{3\alpha}{x} \right)^2 \tag{5.12}$$

and $\varphi_2(x) := \dfrac{d\psi_2}{dx}$. The fractional Euler-Lagrange equation associated to L is

$$\Gamma(1 + 6\alpha) \overset{6\alpha}{x} + a_2 \Gamma(1 + 4\alpha) \overset{4\alpha}{x} + a_1 \Gamma(1 + 2\alpha) \overset{2\alpha}{x} + \varphi_2(x) = 0. \tag{5.13}$$

If $\alpha = \dfrac{1}{2}$, we obtain the equation

$$\dddot{x}(t) + a_2\ddot{x}(t) + a_1\dot{x}(t) + \varphi_2(x(t)) = 0 \tag{5.14}$$

which also does not admit a classical Lagrangean.

3. The homogeneous Bagley-Törvik equation,

$$aD^2x(t) + bD^{3/2}x(t) + cx(t) = 0, \tag{5.15}$$

where $a, b, c \in \mathrm{I\!R}$, $x(0) = 0$, $D^1x(0) = 0$, describe the dynamics of a flat rigid body embedded in a Newton fluid. The equation (5.15) is the fractional differential equation for the Lagrangean function $L_1 : \overset{7/4}{T}\,\mathrm{I\!R} \to \mathrm{I\!R}$ given by

$$L_1\left(x(t), \overset{5/4}{x}(t), \overset{7/4}{x}(t)\right) = \frac{c}{\Gamma(9/4)} - \frac{b\Gamma(3)}{\Gamma(3/2)}\left[\overset{5/4}{x}(t)\right]^{1/2} - \frac{a\Gamma(3)}{\Gamma(3/2)}\left[\overset{7/4}{x}(t)\right]^{1/2}. \tag{5.16}$$

For proof we consider the function $L : \overset{7\alpha}{T}\,\mathrm{I\!R} \to \mathrm{I\!R}$, given by

$$L\left(x, \overset{5\alpha}{x}, \overset{7\alpha}{x}\right) := \frac{c}{\Gamma(2+\alpha)}x^{1+\alpha} - \frac{b_1}{\Gamma(1+2\alpha)}\left(\overset{5\alpha}{x}\right)^{2\alpha} - \frac{a_1}{\Gamma(1+2\alpha)}\left(\overset{7\alpha}{x}\right)^{2\alpha}, \tag{5.17}$$

where $a_1 = a\Gamma(3)$, $b_1 = b\Gamma(5/2)$. The fractional Euler-Lagrange equation associated to (5.17) is the following:

$$D_x^\alpha L - d_t^{5\alpha}\left(D^\alpha_{\overset{5\alpha}{x}} L\right) - d_t^{7\alpha}\left(D^\alpha_{\overset{7\alpha}{x}} L\right) = 0. \tag{5.18}$$

Using the relation (2.3), we have

$$\begin{aligned}
&D_x^\alpha L = \frac{c}{\Gamma(1+\alpha)}D_x^\alpha(x(t))^{1+\alpha} = \frac{cx}{\Gamma(1+\alpha)}\cdot\frac{\Gamma(2+\alpha)}{\Gamma(1)} = cx(t);\\
&D^\alpha_{\overset{5\alpha}{x}} L = -\frac{b_1}{\Gamma(1+2\alpha)}D^\alpha_{\overset{5\alpha}{x}}\left(\overset{5\alpha}{x}(t)\right)^{2\alpha} = -\frac{b_1}{\Gamma(1+\alpha)}\left(\overset{5\alpha}{x}(t)\right)^{\alpha};\\
&D^\alpha_{\overset{7\alpha}{x}} L = -\frac{a_1}{\Gamma(1+2\alpha)}D^\alpha_{\overset{7\alpha}{x}}\left(\overset{7\alpha}{x}(t)\right)^{2\alpha} = -\frac{a_1}{\Gamma(1+\alpha)}\left(\overset{7\alpha}{x}(t)\right)^{\alpha};\\
&d_t^{5\alpha}\left(\overset{\alpha}{D}_{\overset{5\alpha}{x}} L\right) = -b_1\overset{6\alpha}{x}(t);\; d_t^{7\alpha}\left(D^\alpha_{\overset{7\alpha}{x}} L\right) = -a_1\overset{8\alpha}{x}(t).
\end{aligned} \tag{5.19}$$

¿From (5.18), with (5.19), we obtain the equation

$$a_1\overset{8\alpha}{x}(t) + b_1\overset{6\alpha}{x}(t) + cx(t) = 0. \tag{5.20}$$

For $\alpha = \dfrac{1}{4}$, the relation (5.20) leads to the equation

$$a_1\overset{2}{x}(t) + b_1\overset{3/2}{x}(t) + cx(t) = 0 \tag{5.21}$$

which is equivalent to (5.15). Consequently, (5.15) is the fractional differential equation associated to L_1 given by (5.16).

6 Conclusions

Starting from the idea that the fractional techniques and methods require a fractional geometrical model we consider the fractional tangent bundle of order k. On this tangent bundle are defined some objects and structures having a geometrical character. We also consider fractional Lagrangeans on that bundle.

As the examples show there are some equations which do not come from a variational principle, but they can be described as the Euler–Lagrange equations of fractional Lagrangeans.

Most approaches of the fractional order systems in literature are studied with numerical methods. A logical target for us is to analyse the behaviour of a fractional dynamics including geometrical methods. Related results can be found in [4, 5].

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