# Two parameter singular perturbation problems for sine-Gordon type equations

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Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with $C^1$ boundary $\partial\Omega$. Consider the real Hilbert space $L^2(\Omega)$, endowed with the usual inner product $(u,v)_{L^2(\Omega)} = \displaystyle\int_\Omega u(x)\,v(x)\,dx$ and the norm $|\cdot|$, and the real Sobolev space $H_0^1(\Omega)$, with the inner product $(u,v)_{H_0^1(\Omega)} = \int_\Omega \big(\nabla u(x),\,\nabla v(x)\big)_{\mathbb{R}^n}\,dx$ and the norm $||\cdot||$.

We consider the following initial-boundary problem for sine-Gordon type equation

$$\begin{cases} \varepsilon\,\partial_t^2 u_{\varepsilon\delta} + \delta\,\partial_t u_{\varepsilon\delta} + A u_{\varepsilon\delta} + b\,\sin u_{\varepsilon\delta} = f, \text{ in } Q_T, \\ u_{\varepsilon\delta}(x,0) = u_0(x), \quad \partial_t\, u_{\varepsilon\delta}(x,0) = u_1(x), \quad x \in \Omega, \\ u_{\varepsilon\delta}|_{\partial\Omega} = 0, \quad t \geq 0, \end{cases} \tag{$P_{\epsilon\delta}$}$$

where $T > 0$, $Q_T = \Omega \times (0,T)$, $f \in L^2(Q_T)$, $u_0 \in V = H_0^1(\Omega)$, $u_1 \in H = L^2(\Omega)$, $b \in \mathbb{R}$, $b \neq 0$, $\varepsilon, \delta$ are two small parameters and $A$ is a strong elliptic operator of the type

$$A : D(A) = H^2(\Omega) \cap H_0^1(\Omega) \mapsto L^2(\Omega),\, A\,u = -\sum_{i,j=1}^{n} \partial_{x_i}\big(a_{ij}(x)\,\partial_{x_j} u(x)\big). \tag{1}$$

Namely, we suppose that the following conditions:

$$\textbf{(HA)} \begin{cases} a_{ij} \in L^\infty(\Omega),\; a_{ij}(x) = a_{ji}(x), \text{ a.e. in } \Omega, \\ \omega_0\,|\xi|^2 \leq \displaystyle\sum_{i,j=1}^{n} a_{ij}(x)\,\xi_i\,\xi_j \leq \omega_1\,|\xi|^2, \text{ a.e. in } \Omega,\; \forall \xi \in \mathbb{R}^n,\; 0 < \omega_0 \leq \omega_1; \end{cases}$$

**(HSG)** $q_0 = \omega_0 - \lambda_1^{-1}\,|b| > 0$, where $\lambda_1$ is the first eigenvalue of the spectral problem $-\Delta u = u,\ u|_{\partial\Omega} = 0$
are fulfilled.

We investigate the behavior of solutions $u_{\varepsilon\delta}$ to the problem $(P_{\varepsilon\delta})$ in two different cases:
$(i)$ $\varepsilon \to 0$ and $\delta \geq \delta_0 > 0$, relative to the solutions to the following unperturbed

system:

$$\begin{cases} \delta\, \partial_t l_\delta(x,t) + A l_\delta(x,t) + b \sin l_\delta(x,t) = f(x,t), \quad (x,t) \in Q_T, \\ l_\delta(x,0) = u_0(x), \quad x \in \Omega, \\ l_\delta|_{\partial\Omega} = 0, \quad t \geq 0; \end{cases} \tag{$P_\delta$}$$

$(ii)$ $\varepsilon \to 0$ and $\delta \to 0$, relative to the solutions to the following unperturbed system:

$$\begin{cases} Av(x,t) + b\ \sin v(x,t) = f(x,t), \quad (x,t) \in Q_T, \\ v|_{\partial\Omega} = 0, \quad t \geq 0. \end{cases} \tag{$P_0$}$$

We obtain some *a priori* estimates of solutions to the perturbed problem, which are uniform with respect to parameters, and a relationship between solutions to both problems. We establish that the solution to the unperturbed problem has a singular behavior, relative to the parameters, in the neighbourhood of $t = 0$.

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