On free nilpotent commutative automorphic loops

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In [4], there was announced (see in [5] her presentation) about one correspondence between the class of all commutative rings with a unit and one finitely axiomatizable class of loops. Moreover, the correspondence by the ring of integers corresponds to a metabelian commutative free automorphic loop of rank two. We derive some equational properties of this loop and prove, that the elementary theory free n-nilpotent commutative and non-associative automorphic loop are recursively undecidable.

1. As is well known (see [1]), the inner mapping group $J(L)$ of loop L is the group generated by all mappings of the form $T_a = R_a L_a^{-1}$, $L_{a,b} = L_b L_a L_{ab}^{-1}$, $R_{a,b} = R_a R_b R_{ab}^{-1}$, where $aL_b = bR_a = ab$ $(a, b \in L)$.

The subloop H of the loop L is normal in L if one of the following two equivalent conditions is satisfied:

(i) $H \cdot xy = Yx \cdot y$, $xy \cdot H = x \cdot yH$, $xh = Hx$ for any $x, y \in L$;

(ii) $H\alpha = H$ for any $\alpha \in J(L)$.

A loop L is said to be an automorphic loop (or A-loop) if any inner permutation from $J(L)$ is an automorphism on L [1].

2. The equational theory of automorphic loops was studied in [3]. Here the left associator and right associator $[a, b, c]$ of any three elements (a, b, c) of any three elements of the loop, and also the commutator $[a, b]$ of any two elements of the loop by the following equalities:

$$(a, b, c) = a\backslash((ab \cdot c)/(bc)), \quad [a, b, c] = ((ab)\backslash(a \cdot bc))/c, \ [a, b] = a\backslash(b\backslash(ab)).$$

Then the internal mappings of the form $L_{a,b}$, $R_{a,b}$, T_a, of the loop L are expressed in terms of associator and commutator formulas

$$c\,L_{a,b} = c(c, a, b), \ \ cR_{a,b} = [a, b, c]c, \ \ aT_b = a \cdot a[a, b].$$

Hence it follows that the class of all automorphic loops is a variety defined by the identities in the class of all loops:

$$xy \cdot [xy, z] = x[x, z] \cdot y[y, z];$$

$$xy \cdot (xy, z, t) = x(x, z, t) \cdot y(y, z, t);$$

$$[x, y, zt] \cdot zt = [x, y, z]z \cdot [x, y, t]t.$$

The subloop L' of the loop L generated by all the associators and commutators is called the associant-commutant of the loop L.

Let L be a free automorphic loop of rank $r \geq 2$ and $L_{(n)}$, $1 \leq n < \omega$ subloop of the loop L defined inductively: $L_{(1)} = L'$, $L_{(n)}$ for the $1 < n < \omega$ – is a subloops generated in a loop L by all associators and commutators of the form (x, y, z), $[y, z, x]$, $[x, y]$, where $x \in L_{(n-1)}$, $y, z \in L$. Obviously, all the subloops $L_{(n)}$ are closed with respect to the inner permutations of the group $J(L)$, therefore they are normal subloops. Loops isomorphic factor-loops $L/L_{(n)}$ of a free automorphic loop L by its normal subloop $L_{(n)}$ are called free automorphic loops nilpotent of class n. Loops

$L/L_{(2)}$, as well as any nonassociative or noncommutative loop, is isomorphic to the homomorphic image of one of these loops (as in groups) we call metabelian automorphic loops. Now it is easy to prove

Proposition 1. *In the class of all loops the basis of identities of a free metabelian automorphic loop* $L/L_{(2)}$ *consists of the following:*

$$[xy, z] = [x, z][y, z], \quad (xy, z, t) = (x, z, t)(y, z, t), \quad (x, y, zt) = (x, y, z)(x, y, t).$$

It follows from [3] that metabelian loops possess the following equational properties, which we present in the following two sentences.

Proposition 2. *In any metabelian loop, the following commutator identities are true:*

$$[y, x]^{-1} = [y^{-1}, x], \quad [x/y, z] = [y\backslash x, z] = [x \cdot y^{-1}, z],$$

$$[x\backslash y, z] = [x, z\backslash y] = [xy^{-1}, y], \quad [x^m, y] = [x, y]^m \text{ for any integer } m.$$

Proposition 3. *In any metabelian loop, associative identities are true:*

$$(x, y, z) = (y, x, z)(x, z, y), \quad (x, y, z) = (z, y, x)^{-1},$$

$$(x, y, z)^{-1} = (x^{-1}, y, z), \quad (x^m, y, z) = (x, y, z)^m \text{ for any integer } m.$$

3. In [4], a correspondence is established between commutative rings with a unit and metabelian loops of a singularly finitely axiomatizable class. In particular, the ring of integers Z corresponds to a free metabelian commutative loop of the second rank. An effective method is given that allows for each closed formula φ, the signature language of a ring with unity, obtain the formula φ_1 for commutative loops, such that the truth φ on the ring Z is equivalent to the truth φ_1 on the loop F_2. Since the elementary theory of the ring of integers is unsolvable (see [2]), we immediately obtain

Lemma 6. *The elementary theory of a free metabelian commutative* auto-morphic *loop* F_2 *of rank 2 is undecidable.*

Naturally, a more general

Lemma 7. *Elementary theory of a free metabelian commutative* auto-morphic *loop* F_n *with* n *free generators for* $n \geq 3$ *undecidable.*

Using the induction method, on the basis of Lemma 1, 2 and Propositions $1 - 3$, we arrive at the conclusion ehat the following is true:

Theorem 1. *The elementary theory of a free* k*-nilpotent commutative automorphic loop* F_n *with* n *free generators is undecidable for* $n \geq 2$, $k \geq 2$.

Bibliography

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