Definition and example of n-ary Moufang loop

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Summary. In this work necessary and sufficient conditions that isotope of n-IP-loop ($n \in N^*$, $n > 3$) is also n-IP-loop are proved. Definition of n-ary Moufang loop is given, example of such loop is constructed.
Keywords: n-IP-quasigroup, n-IP-loop, Moufang loop, isotopy, LP-isotopy.
Main concepts and definitions. Quasigroup $Q(A)$ of arity n, $n \geq 2$, is called an n-IP-quasigroup if there exist permutations ν_{ij}, $i,j \in \overline{1,n}$ of the set Q such that the following identities are true:

$$A(\{\nu_{ij}x_j\}_{j=1}^{i-1}, A(x_1^n), \{\nu_{ij}x_j\}_{j=i+1}^{n}) = x_i$$

for all $x_1^n \in Q^n$, where $\nu_{ii} = \nu_{i\,n+1} = \varepsilon$, ε denotes identity permutation of the set Q [1]. The matrix

$$\left[\nu_{ij}\right] = \begin{bmatrix} \varepsilon & \nu_{12} & \nu_{13} & \dots & \nu_{1n} & \varepsilon \\ \nu_{21} & \varepsilon & \nu_{23} & \dots & \nu_{2n} & \varepsilon \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \nu_{31} & \nu_{32} & \varepsilon & \dots & \varepsilon & \varepsilon \end{bmatrix}$$

is called *an inversion matrix* for an n-IP-quasigroup, the permutations $\nu_{i,j}$ are called *inversion permutations.* An element e is called a unit of n-ary operation $Q()$, if the following equality is true $(\overset{i-1}{e}, x, \overset{n-i}{e}) = x$ for all $x \in Q$ and $i \in \overline{1,n}$. n-Ary quasigroup with unit element is called an n-ary loop [1].

Permutations I_{ij} of the set Q are defined by equities

$$(\overset{i-1}{e}, x, \overset{j-i-1}{e}, I_{ij}x, \overset{n-j}{e}) = e$$

for all $x \in Q$ and $i, j \in \overline{1, n}$.

N-ary quasigroup (Q, B) is isotopic to n-ary quasigroup (Q, A) (the number n is the same in both quasigroups) if there exist a tuple of permutations $T = (\alpha_1^{n+1})$ of the set Q such that

$$B(x_1^n) = \alpha_{n+1}^{-1} A(\alpha_1 x_1, \ldots, \alpha_n x_n).$$

In this case we write $B = A^T$ [1]. Isotope of the form $T = (\alpha_1^n, \varepsilon)$ is called main isotope.

As usual $\overline{a} = a_1^n$. Main isotope is called LP-isotope, if $\alpha_i = L_i^{-1}(\overline{a})$ for all $i \in \overline{1, n}$, where $L_i(\overline{a})x = A(a_1^{i-1}, x, a_{i+1}^n)$.

LP-isotope of a quasigroup (Q, A) is a loop with unit $f = A(a_1^n)$ [1]. If $A = B$, then the tuple T is called autotopy of n-ary quasigroup A.

Main results

Theorem 1. *LP-isotope $B = A^{(L_1^{-1}(\overline{a}), L_2^{-1}(\overline{a}), \ldots, L_n^{-1}(\overline{a}), \varepsilon)}$ of n-IP-loop A with unit e is an n-IP-loop if and only if*

$$T_i = (\{I_{ij}^e L_j^{-1}(\overline{a}) I_{ij}^f L_j(\overline{a})\}_{j=1}^{i-1}, L_i(\overline{a}), \{I_{ij}^e L_j^{-1}(\overline{a}) I_{ij}^f L_j(\overline{a})\}_{j=i+1}^{n}, L_i^{-1}(\overline{a})),$$

$i \in \overline{1, n}$, are autotopies of n-IP-loop (Q, A) for any fixed element $a \in Q$, where $[I_{ij}^e]$ is inversion matrix for (Q, A), $[I_{ij}^f]$ is inversion matrix for (Q, B) [2].

If $n = 3$, then we obtain results from [3], if $n = 2$, then autotopies T_i are transformed into the well known Moufang identities.

Definition 1. *n-Loop (Q, A) $(n > 2)$ is called n-ary Moufang loop, if any its LP-isotope is an n-IP-loop.*

In contrast to the binary case, for $n > 2$ a Moufang loop is not an IP-loop [1]. If (Q, A) is an n-IP-loop, then we call it n-IP Moufang loop.

Example 4. *Let $A(x_1^4) = (x_1^4) = x_1 \cdot x_2 \cdot x_3 \cdot x_4$ be a 4-IP-quasigroup which is defined over a binary Abelian group $(Q, \cdot)$.*

It is possible to check that (Q, A) is a 4-IP-loop with unit e that coincides with the unit of the group $(Q, \cdot)$ and with invertible matrix $[I_{ij}^e]$. We suppose that $I_{ij}^e L_j^{-1}(\overline{a}) I_{ij}^f L_j(\overline{a}) = \varphi_{ij}(\overline{a})$.

Then autotopy T_1 from Theorem 1 is transformed into the following identity:

$$(((x_1, a_2, a_3, a_4), x_2, x_3, x_4), a_2, a_3, a_4) = (x_1, \varphi_{12}(\overline{a})x_2, \varphi_{13}(\overline{a})x_3, \varphi_{14}(\overline{a})x_4).$$

This identity is true if we take $\varphi_{12}(\overline{a})x = a_2 \cdot a_3 \cdot a_4 \cdot x$, $\varphi_{13}(\overline{a})x = x$, $\varphi_{14}(\overline{a})x = x \cdot a_2 \cdot a_3 \cdot a_4$.

Similarly we see that permutations T_2, T_3, T_4 are autotopies of loop (Q, A). Therefore (Q, A) is a symmetric 4-IP-Moufang-loop.

In the similar way it is possible to construct an n-IP-Moufang-loop of any arity n.

Bibliography

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