On the algebraic properties of the ring of Dirichlet convolutions

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Let $(\Gamma,\cdot)$ be a commutative monoid of finite type and let R be a commutative ring with unity. In the set of functions defined on Γ with values in R, $\mathcal{F}(\Gamma,R) := \{\alpha : \Gamma \to R\}$, we consider the operations

$$(\alpha+\beta)(n) := \alpha(n)+\beta(n),\ (\forall)n\in\Gamma \text{ and } (\alpha\cdot\beta)(n) := \sum_{ab=n}\alpha(a)\beta(b),\ (\forall)n\in\Gamma.$$

It is well known that $(\mathcal{F}(\Gamma,R),+,\cdot)$ is a commutative ring with unity, see for instance [3]. Let M be a R-module. Let $\mathcal{F}(\Gamma,M) := \{f : \Gamma \to M\}$. For any $f,g\in\mathcal{F}(\Gamma,M)$ and $\alpha\in\mathcal{F}(\Gamma,R)$, we define:

$$(f+g)(n) := f(n)+g(n),\ (\forall)n\in\Gamma \ \text{ and } (\alpha\cdot f)(n) := \sum_{ab=n}\alpha(a)f(b),\ (\forall)n\in\Gamma.$$

We show that $\mathcal{F}(\Gamma,M)$ has a natural structure of a $\mathcal{F}(\Gamma,R)$-module and we discuss certain properties of the functor $M\mapsto(\Gamma,M)$. Given a morphism of monoids $L:(\Gamma,\cdot)\to(M,+)$, we show that the induced map $\Phi_{L,M}:\mathcal{F}(\Gamma,M)\to\mathcal{F}_L(\Gamma,M),\ \Phi_{L,M}(f)(n) := L(n)f(n)$ is a morphism of $\mathcal{F}(\Gamma,R)$-modules.
Let A be a commutative ring with unity and let $i:A\to R$ be a morphism of rings with unity. Let M be an R-module. An A-derivation $D:R\to M$ is an A-linear map, satisfying the Leibniz rule, i.e. $D(fg)=fD(g)+gD(f), (\forall)f,g\in R$. The set of A-derivations $\mathrm{Der}_A(R,M)$ has a natural structure of a R-module. Let $D\in\mathrm{Der}_A(R,M)$ and let $\delta:\Gamma\to(M,+)$ be a morphism of monoids. We prove that

$$\widetilde{D}:\mathcal{F}(\Gamma,R)\to\mathcal{F}(\Gamma,M),\ \widetilde{D}(\alpha)(n)=D(\alpha(n))+\alpha(n)\delta(n),\ (\forall)n\in\Gamma,\ \text{is an } A\text{-derivation.}$$

Assume that the monoid $(\Gamma,\cdot)$ is cancellative, i.e. $xy=xz$ implies $y=z$. Let $G(\Gamma)$ be the Grothendieck group associated to Γ, see [2]. In the set

$$\mathcal{F}^f(G(\Gamma),R) := \{\alpha : G(\Gamma)\to R\ :\ (\exists)d\in\Gamma \text{ such that } (\forall)q\in G(\Gamma),\alpha(q)\neq 0\Rightarrow dq\in\Gamma\}.$$

we consider the operations

$$(\alpha+\beta)(q) := \alpha(q)+\beta(q),\ (\forall)q\in G(\Gamma) \text{ and } (\alpha\cdot\beta)(q) := \sum_{q'q''=q}\alpha(q')\beta(q''),\ (\forall)q\in G(\Gamma).$$

We prove that $\mathcal{F}^f(G(\Gamma),R)$ is an extension of the ring $\mathcal{F}(\Gamma,R)$. Let M be an R-module. We consider the set

$$\mathcal{F}^f(G(\Gamma),M) := \{f : G(\Gamma)\to M\ :\ (\exists)d\in\Gamma \text{ such that } (\forall)q\in G(\Gamma),f(q)\neq 0\Rightarrow dq\in\Gamma\}.$$

For $f,q\in\mathcal{F}^f(G(\Gamma),M)$ we define $(f+g)(q) := f(q)+g(q),(\forall)q\in G(M)$. Given $\alpha\in\mathcal{F}^f(G(\Gamma),R)$ and $f\in\mathcal{F}^f(G(\Gamma),M)$ we define $(\alpha\cdot f)(q) := \sum_{q'q''=q}\alpha(q')f(q''),\ (\forall)q\in G(\Gamma)$. We prove that $\mathcal{F}^f(G(\Gamma),M)$ has a structure of an $\mathcal{F}^f(G(\Gamma),R)$-module and we study the connections between

the associations $M \mapsto \mathcal{F}(\Gamma, M)$ and $M \mapsto \mathcal{F}^f(\Gamma, M)$.
In particular, we show that if $D \in \mathrm{Der}_A(R, M)$ is an A-derivation and $\delta : \Gamma \to (M, +)$ is a morphism of monoids, then we can construct an A-derivation on $\overline{D} : \mathcal{F}^f(G(\Gamma), R) \to \mathcal{F}^f(G(\Gamma), M)$ which extend $\widetilde{D}$.
The most important case, largely studied in analytic number theory [1], is the case when R is a domain (or even more particullary, when $R = \mathbb{C}$) and $\Gamma = \mathbb{N}^*$ is the multiplicative monoid of positive integers. Cashwell and Everett showed in [4] that $\mathcal{F}(\mathbb{N}^*, R)$ is also a domain. Moreover, if R is an UFD with the property that $R[[x_1, \ldots, x_n]]$ are UFD for any $n \geq 1$, then $\mathcal{F}(\mathbb{N}^*, R)$ is also an UFD, see [5]. It is well known that the Grothendieck group associated to $\mathbb{N}^*$ is $\mathbb{Q}_+^* :=$ the group of positive rational numbers. We show that

$$\mathcal{F}^f(\mathbb{Q}_+^*, R) \cong R[[x_1, x_2, \ldots]][x_1^{-1}, x_2^{-1}, \ldots],$$

and, in particular, if $R[[x_1, x_2, \ldots]]$ is UFD, then $\mathcal{F}^f(\mathbb{Q}_+^*, R)$ is an UFD.
We make some remarks in the following case: Let $U \subset \mathbb{C}$ be an open set and let $\mathcal{O}(U)$ be the ring of holomorphic functions defined in U with values in $\mathbb{C}$. It is well known that $\mathcal{O}(U)$ is a domain. We consider

$$\widetilde{D} : \mathcal{F}(\mathbb{N}^*, \mathcal{O}(U)) \to \mathcal{F}(\mathbb{N}^*, \mathcal{O}(U)),\ \widetilde{D}(\alpha)(n)(z) := \alpha(n)'(z) - \alpha(n)(z)\log n,\ (\forall) n \in \mathbb{N}, z \in U.$$

We note that $\widetilde{D}$ is a $\mathbb{C}$-derivation on $\mathcal{F}(\mathbb{N}^*, \mathcal{O}(U))$. Assume that the series of functions $F_\alpha(z) := \sum_{n=1}^{\infty} \frac{\alpha(n)(z)}{n^z}$, $z \in U$, and and $G_\alpha(z) = \sum_{n=1}^{\infty} \left(\frac{\alpha(n)(z)}{n^z}\right)'$, $z \in U$, are uniformly convergent on the compact subsets $K \subset U$. It is well known that, in this case, F defines a derivable (holomorphic) function on U and, moreover, $F' = G$. It is easy to see that $F'_\alpha = F_{\widetilde{D}(\alpha)}$. Further connections between the $\mathbb{C}$-linear independence of $\alpha_1, \ldots, \alpha_m \in \mathcal{F}(\mathbb{N}^*, \mathcal{O}(U))$, with some suplimentary conditions, and their associated series $F_{\alpha_1}, \ldots, F_{\alpha_m}$ were made in [6].

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