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ABOUT THE VALUES OF THE PARAMETERS THAT DESCRIBE THE NEWTONIAN PROBLEM OF THE EIGHT BODIES

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Abstract. We consider the Newtonian restricted eight bodies problem with incomplete symmetry. We investigate the stability of some configurations of this problem. The values of the parameters that describe the Newtonian problem of the eight bodies are investigate by some numerical methods. For geometric parameter the intervals of linear stability and instability are found. All relevant and numerical calculation are done with the computer algebra system Mathematica.

Keywords: Newtonian problem; differential equation of motion; configuration; stationary points; linear stability.

CONSIDERAŢII DESPRE VALORILE PARAMETRILOR CE DESCRIU PROBLEMA NEWTONIANĂ A CELOR OPT CORPURI

Rezumat. Se consideră problema newtoniană, mărginită a opt corpuri cu simetrie incompletă. Se cercetează stabilitatea unor configuraţii ale acestei probleme. Folosind metode numerice se determină valorile parametrilor care descriu problema cercetată a celor opt corpuri. Pentru parametrul geometric se determină intervalele de stabilitate şi instabilitate liniară. Toate calculele numerice se obţin aplicând sistemul de calcul algebric computerizat Mathematica.

Cuvinte cheie: problema Newtoniană, ecuaţia diferenţială a mişcării, configuraţie, puncte staţionare, stabilitate lineară.

Introduction

At present, qualitative studies of dynamical models of space are based on the search for exact particular solutions of differential equations of motion and subsequent analysis of their stability using the latest advances in computer mathematics. For this, it is required, first of all, to develop mathematical methods and algorithms for constructing exact partial solutions, since in the case of the Newtonian many bodies problem, for example, the number of solutions found is very limited.

Most of the exact solutions found for the Newtonian n-bodies problem belong to the class of so-called homographic solutions, the sufficient conditions for their existence were obtained by A. Wintner in the first half of the twentieth century, and the necessary conditions were formulated later by E. A. Grebenikov (see [3]).

The research method is based on the application of the analytic and qualitative theory of differential equations, the stability theory of Lyapunov-Poincaré, and also on the use of the capabilities of modern computer algebra systems for performing numerical calculations, processing symbolic information, and visualizing the obtained results.

It is known that in studying of the differential equations of restricted problems, first of all, it is necessary to study the existence of particular solutions of „equilibrium positions” in the unlimited small size problems.

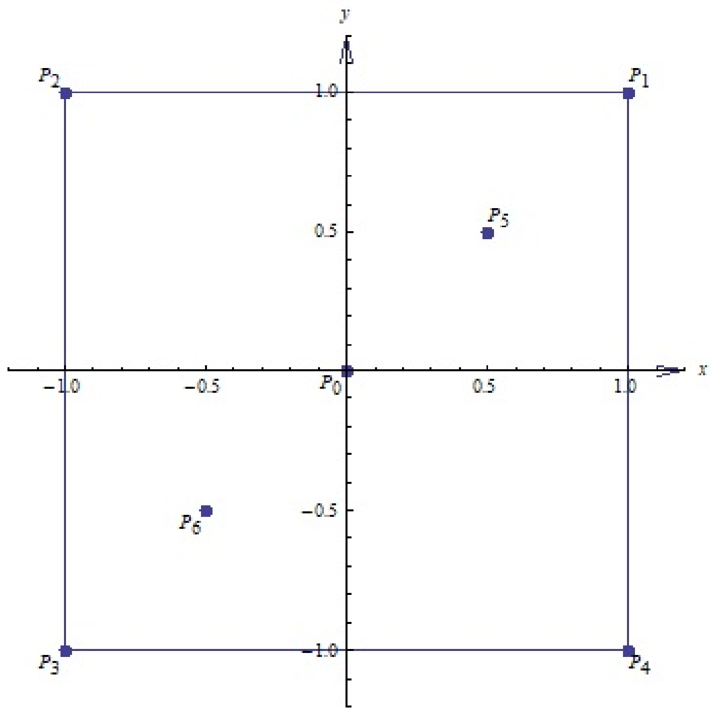

Figure 1. Studied model

Description of the configuration

We will study a particular case of the n-bodies problem describing in a non-inertial space P_0xyz the motion of seven bodies P_0, P_1, P_2, P_3, P_4, P_5, P_6, with the masses m_0, m_1, m_2, m_3, m_4, m_5, m_6, which attract each other in accordance with the law of universal attraction. We will investigate the planar dynamic pattern formed by a square in the vertices of which the points P_1, P_2, P_3, P_4, are located, the other two points P_5, P_6, having the masses $m_5 = m_6$ are on the diagonal P_1P_3 of the square at equal distances from point P_0, in around which this configuration rotates with a constant angular velocity ω which is determined from the model parameters (see $Fig.1$).

The differential equations of the Newtonian problem of seven bodies in a non-inertial cartesian coordinate system P_0xyz have the form:

$$\begin{cases} \frac{d^2x_k}{dt^2} + \frac{f(m_0+m_k)x_k}{r_k^3} = \frac{\partial R_k^*}{\partial x_k}, \\ \frac{d^2y_k}{dt^2} + \frac{f(m_0+m_k)y_k}{r_k^3} = \frac{\partial R_k^*}{\partial y_k}, \\ \frac{d^2z_k}{dt^2} + \frac{f(m_0+m_k)z_k}{r_k^3} = \frac{\partial R_k^*}{\partial z_k}, \\ k = 1, 2, ..., 6; \end{cases} \tag{1}$$

where $R_k^*(k = 1, 2, ..., 6)$ are the perturbation functions which are expressed by the relations:

$$\begin{cases} R_k^* = f\sum_{j=1}^{6} m_j \left(\frac{1}{\Delta_{kj}} - \frac{x_kx_j+y_ky_j+z_kz_j}{r_j^3} \right), \ j \neq k, \\ \Delta_{kj}^2 = (x_j - x_k)^2 + (y_j - y_k)^2 + (z_j - z_k)^2, \\ r_j^2 = x_j^2 + y_j^2 + z_j^2, \\ k = 1, 2, ..., 6. \end{cases} \tag{2}$$

To determine ω we will carry out coordinate transformation that would exclude from the right-hand sites of the equations (1) the time t:

$$\begin{cases} x_j = X_j \cos(\omega t) - Y_j \sin(\omega t), \\ y_j = X_j \sin(\omega t) + Y_j \cos(\omega t), \\ z_j = Z_j. \end{cases} \tag{3}$$

Since we study the planar configuration, we have $z_j = 0$, $j = 0, 1, ..., 6$. In the new coordinates the equations (1), have the form:

$$\begin{cases} \frac{d^2 X_k}{dt^2} = \omega^2 X_k + 2\omega \frac{dY_k}{dt} - \frac{f(m_0+m_k)X_k}{r_k^3} + \frac{\partial R_k^*}{\partial X_k}, \\ \frac{d^2 Y_k}{dt^2} = \omega^2 Y_k - 2\omega \frac{dX_k}{dt} - \frac{f(m_0+m_k)Y_k}{r_k^3} + \frac{\partial R_k^*}{\partial X_k}, \end{cases} \tag{4}$$

$$\begin{cases} R_k^* = f \sum\limits_{j=1}^{6} m_j \left(\frac{1}{\Delta_{kj}} - \frac{X_k X_j + Y_k Y_j}{r_j^3} \right), \ j \neq k, \\ \Delta_{kj}^2 = (X_j - X_k)^2 + (Y_j - Y_k)^2, \\ r_j^2 = X_j^2 + Y_j^2, \\ k = 1, 2, ..., 6. \end{cases} \tag{5}$$

We can assume that $P_1(1,1)$, $P_2(-1,1)$, $P_3(-1,-1)$, $P_4(1,-1)$, $P_5(\alpha,\alpha)$, $P_6(-\alpha,-\alpha)$, $f = 1, m_0 = 1$, $m_5 = m_6$.

Then out of the differential equations of the motion we obtain the existence conditions of this configuration:

$$\begin{cases} m_1 = m_3, \ m_2 = m_4 = f_1(m_1, \alpha), \\ m_5 = m_6 = f_2(m_1, \alpha), \ \omega^2 = f_3(m_1, \alpha). \end{cases} \tag{6}$$

Intervals of admissible values for the parameter α are determined by the conditions

$$m_2 = m_4 > 0; m_5 = m_6 > 0; \omega^2 > 0. \tag{7}$$

The functional dependences $m_2 = m_4 = f_1(m_1, \alpha)$, $m_5 = m_6 = f_2(m_1, \alpha,)$, $\omega^2 = f_3(m_1, \alpha)$ can be seen from graphs obtained with the graphical package of Mathematica (see Fig. 2–4).

Theorem 1. *The verification of relations* (7) *represents the sufficient condition of existence of the homographic solution of the Newtonian problem of seven bodies, the its configuration of which represents a square $P_1P_2P_3P_4$ with one of the bodies (P_0) located in the origin of the coordinates, and the other two P_5, P_6 are located of the diagonal P_1P_3.*

For example, for $m_1 = 0.1$ this dependence is shown in Fig. 5.

In the Table 1 are displayed the admissible intervals of α according to some values of m_1 , approximately calculated using the graphical tools of Mathematica.

It is known that this dynamic model generates a new problem - the restricted problem of eight bodies. It will be studied the motion of the body P with a infinitely small mass (the so-called passive gravitational body) in the gravitational field by the given seven bodies.

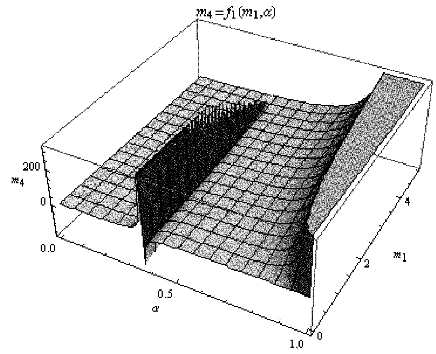

Figure 2. The functional dependences $m_2 = m_4 = f_1(m_1, \alpha)$

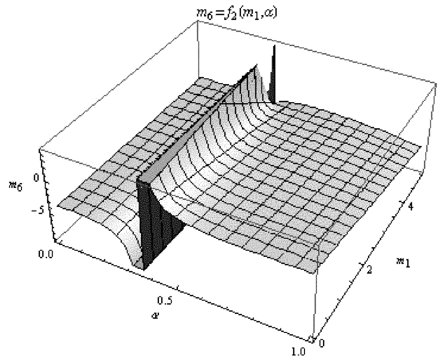

Figure 3. The functional dependences $m_5 = m_6 = f_2(m_1, \alpha)$

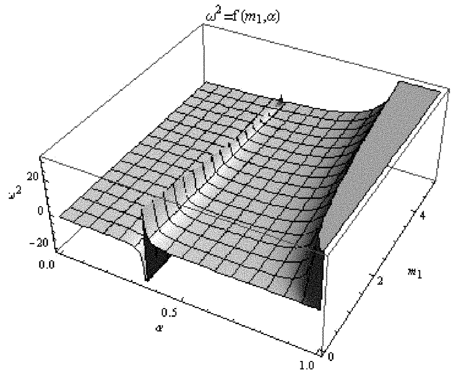

Figure 4. The functional dependences $\omega^2 = f_3(m_1, \alpha)$

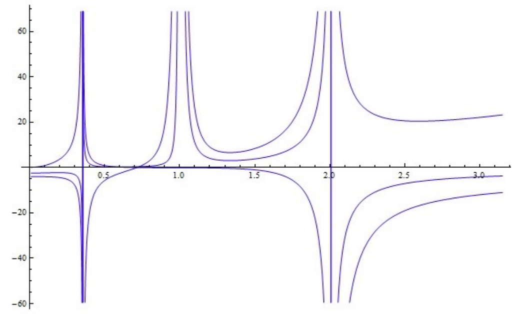

Figure 5. The functional dependences $m_2 = m_4 = f_1(0.1, m_1)$**,** $m_5 = m_6 = f_2(0.1, m_1)$**,** $\omega^2 = f_3(0.1, m_1)$

Table 1:

m_1	Intervals allowed for α
0.0001	—————————
0.001	—————————
0.01	$(0.8582; 0.85857)$
0.1	$(0.715; 0.718)$
1	$(0.48965; 0.5053)$
10	$(0.291; 0.320)$
100	$(0.149; 0.2871)$
1000	$(0.050; 0.2838)$

Differential equations that describe motion of the body $P(x; y; z)$ which gravitates passively in the field of the other seven bodies in the rotating space have the form (see [2]):

$$\begin{cases} \dfrac{d^2X}{dt^2} - 2\omega\dfrac{dY}{dt} = \omega^2 X - \dfrac{fm_0X}{r^3} + \dfrac{\partial R}{\partial X}, \\ \dfrac{d^2Y}{dt^2} + 2\omega\dfrac{dX}{dt} = \omega^2 Y - \dfrac{fm_0Y}{r^3} + \dfrac{\partial R}{\partial Y}, \\ \dfrac{d^2Z}{dt^2} = -\dfrac{fm_0Z}{r^3} + \dfrac{\partial R}{\partial Z}, \end{cases} \tag{8}$$

where

$$\begin{cases} R = f\sum_{j=1}^{6} m_j \left(\dfrac{1}{\Delta_{kj}} - \dfrac{XX_j + YY_j + ZZ_j}{r_j^3}\right), \\ \Delta_j^2 = (X_j - X)^2 + (Y_j - Y)^2 + (Z_j - Z)^2, \\ r_j^2 = X_j^2 + Y_j^2 + Z_j^2,\ r^2 = X^2 + Y^2 + Z^2, \\ j = 1, 2, ..., 6, \end{cases} \tag{9}$$

$(X_j; Y_j; Z_j = 0)$ are the respective coordinates of the bodies P_1, P_2, P_3, P_4, P_5, P_6 and are determined by the conditions of existence of the studied configuration.

Determination of stationary points

According to the definition of the stationary solutions of the differential equations, the equilibrium positions (in case when they exist) are solutions of the functional system of equations:

$$\begin{cases} \dfrac{dx}{dt} = u,\ \dfrac{dy}{dt} = v,\ \dfrac{dz}{dt} = w, \\ \dfrac{du}{dt} = \omega^2 x + 2\omega v - \dfrac{fm_0x}{r^3} + \dfrac{\partial R}{\partial x}, \\ \dfrac{dv}{dt} = \omega^2 y - 2\omega u - \dfrac{fm_0y}{r^3} + \dfrac{\partial R}{\partial y}, \\ \dfrac{dw}{dt} = \dfrac{\partial R}{\partial z}, \end{cases} \tag{10}$$

and

$$\begin{cases} u = 0,\ v = 0,\ w = 0, \\ \omega^2 x + 2\omega v - \dfrac{fm_0x}{r^3} + \dfrac{\partial R}{\partial x} = 0, \\ \omega^2 y - 2\omega u - \dfrac{fm_0y}{r^3} + \dfrac{\partial R}{\partial y} = \dfrac{\partial R}{\partial z} = 0, \end{cases} \tag{11}$$

For simplicity as above it has been taken $f = 1$, $m_0 = 1$. Replacing in relations (11) (X_j, Y_j, Z_j), $m_2 = m_4 = f_1(m_1, \alpha)$, $m_5 = m_6 = f_2(m_1, \alpha)$ and $\omega^2 = f_3(m_1, \alpha)$, determined above for admissible α and m_1 , we obtain the following system:

$$
\begin{cases}
u=0,\ v=0,\ w=0,\\
f(x,y)=\omega^2 x+2\omega v-\dfrac{x}{(x^2+y^2)^{\frac{3}{2}}}+m_1\left(\dfrac{-1-x}{\left((1+x)^2+(1+y)^2\right)^{3/2}}+\dfrac{1-x}{\left((1-x)^2+(1-y)^2\right)^{3/2}}\right)+\\
+m_4\left(\dfrac{1-x}{\left((1-x)^2+(1+y)^2\right)^{3/2}}+\dfrac{-1-x}{\left((1+x)^2+(1-y)^2\right)^{3/2}}\right)+\\
+m_6\left(\dfrac{-\alpha-x}{\left((\alpha+x)^2+(\alpha+y)^2\right)^{3/2}}+\dfrac{\alpha-x}{\left((\alpha-x)^2+(\alpha-y)^2\right)^{3/2}}\right)=0,\\
g(x,y)=\omega^2 y-2\omega u-\dfrac{y}{(x^2+y^2)^{\frac{3}{2}}}+m_1\left(\dfrac{-1-y}{\left((1+x)^2+(1+y)^2\right)^{3/2}}+\dfrac{1-y}{\left((1-x)^2+(1-y)^2\right)^{3/2}}\right)+\\
+m_4\left(\dfrac{1-y}{\left((1-x)^2+(1+y)^2\right)^{3/2}}+\dfrac{-1-y}{\left((1+x)^2+(1-y)^2\right)^{3/2}}\right)+\\
+m_6\left(\dfrac{-\alpha-y}{\left((\alpha+x)^2+(\alpha+y)^2\right)^{3/2}}+\dfrac{\alpha-y}{\left((\alpha-x)^2+(\alpha-y)^2\right)^{3/2}}\right)=0,
\end{cases}
\tag{12}
$$

The equations in the system (12) have a rather complicated structure. Its solving is quite cumbersome. If the solution of the system (12) will to be determined, then it would be obtained the solution of the equilibrium position of differential equations describing the restricted problem of the eight bodies. Using the graphical package of Mathematica for different parameter values α and m_1 have been constructed the graphs of the curves $f(x,y)$ and $g(x,y)$ described by the equations in the system (12). Obviously, the points of intersection of these curves in the plan P_0xy will be the equilibrium positions of the investigated system. For concrete values of m_1 and α we obtain concrete stationary points.

For this we use the program 1.

Program 1

$graph[n\_, a\_]:=$
$Module[\{m_1 = n,\ \alpha = a\},\ gf = f(x,y,m_1,\alpha); gg = g(x,y,m_1,\alpha);$
$cpx = ContourPlot[gf, \{x,\ -2.5,\ 2.5\},\ \{y,\ 2.5,\ 2.5\},\ Contours \to \{0\},$
$ContourShading \to False, PlotPoints \to 100, ContourStyle \to \{Black\},$
$Axes \to True, Frame \to False];$
$cpy = ContourPlot[g, \{x,\ -2.5,\ 2.5\},\ \{y,\ -2.5,\ 2.5\},\ Contours \to \{0\},$
$ContourShading \to False, PlotPoints \to 100, ContourStyle \to \{Dashed\},$
$Axes \to True, Frame \to False];$
$square =\ ListPlot[\{\{1,1\},\{1,-1\},\{-1,-1\},\{-1,1\}\},$
$PlotStyle \to \{PointSize[0.02]\}];$
$points =\ ListPlot[\{\{\alpha,\ \alpha\},\ \{-\alpha,\ -\alpha\}\},\ PlotStyle \to \{PointSize[0.02]\}];$
$M0 := Graphics[Text[''P_0'', \{-0.15, -0.15\}]];$
$M1 := Graphics[Text[''P_1'', \{0.85, 1.05\}]];$
$M2 := Graphics[Text[''P_2'', \{-0.85, 1.05\}]];$

$M3 := Graphics[Text[''P_3'', \{-0.85, -1.05\}]];$
$M4 := Graphics[Text[''P_4'', \{0.85, -1.05\}]];$
$M5 := Graphics[Text[''P_5'', \{\alpha - 0.1, \alpha - 0.1\}]];$
$M6 := Graphics[Text[''P_6'', \{-\alpha + 0.1, -\alpha + 0.1\}]];$
$f1 = FiindRoot[\{gf == 0, gg == 0\}, \{x, 1\}, \{y, 0\}];$
$S1 := Graphics[Text[''S_1'', \{1.55, -0.25\}]]; Print[''S_1'', f1];$
$f2 = FiindRoot[\{gf == 0, gg == 0\}, \{x, 0\}, \{y, 1\}]; Print[''S_2'', f2];$
$f3 = FiindRoot[\{gf == 0, gg == 0\}, \{x, -1\}, \{y, 0\}]; Print[''S_3'', f3];$
$f4 = FiindRoot[\{gf == 0, gg == 0\}, \{x, 0\}, \{y, 1\}]; Print[''S_4'', f4];$
$f12 = FiindRoot[\{gf == 0, gg == 0\}, \{x, 1\}, \{y, 1.05\}]; Print[''N_1'', f12];$
$f13 = FiindRoot[\{gf == 0, gg == 0\}, \{x, 0.9\}, \{y, 0.9\}]; Print[''N_2'', f13];$
$f21 = FiindRoot[\{gf == 0, gg == 0\}, \{x, 1\}, \{y, -1.05\}]; Print[''N_3'', f21];$
$f22 = FiindRoot[\{gf == 0, gg == 0\}, \{x, 0.9\}, \{y, -0.9\}]; Print[''N_4'', f22];$
$f31 = FiindRoot[\{gf == 0, gg == 0\}, \{x, -1\}, \{y, -1.05\}]; Print[''N_5'', f31];$
$f32 = FiindRoot[\{gf == 0, gg == 0\}, \{x, -0.9\}, \{y, -0.9\}]; Print[''N_6'', f32];$
$f41 = FiindRoot[\{gf == 0, gg == 0\}, \{x, -1\}, \{y, -1.05\}]; Print[''N_7'', f41];$
$f42 = FiindRoot[\{gf == 0, gg == 0\}, \{x, -0.9\}, \{y, 0.9\}]; Print[''N_8'', f42];$
$Show[points, square, cpx, cpy, p1, p2, M0, M1, M2, M3, M4, M5, M6, S1],$
$PlotRange \to \{\{-2, 2\}, \{-2, 2\}\}, DisplayFunction \to \$DisplayFunction,$
$AxesLabel \to \{x, y\}, AspectRatio \to Automatic,$
$PlotLabel \to'' m_1 ='' <> ToString[n];'' \alpha ='' <> ToString[a]''''']].$

For m_1=0.01 and α=0.8584 the result of this program is displayed in Figure.6.

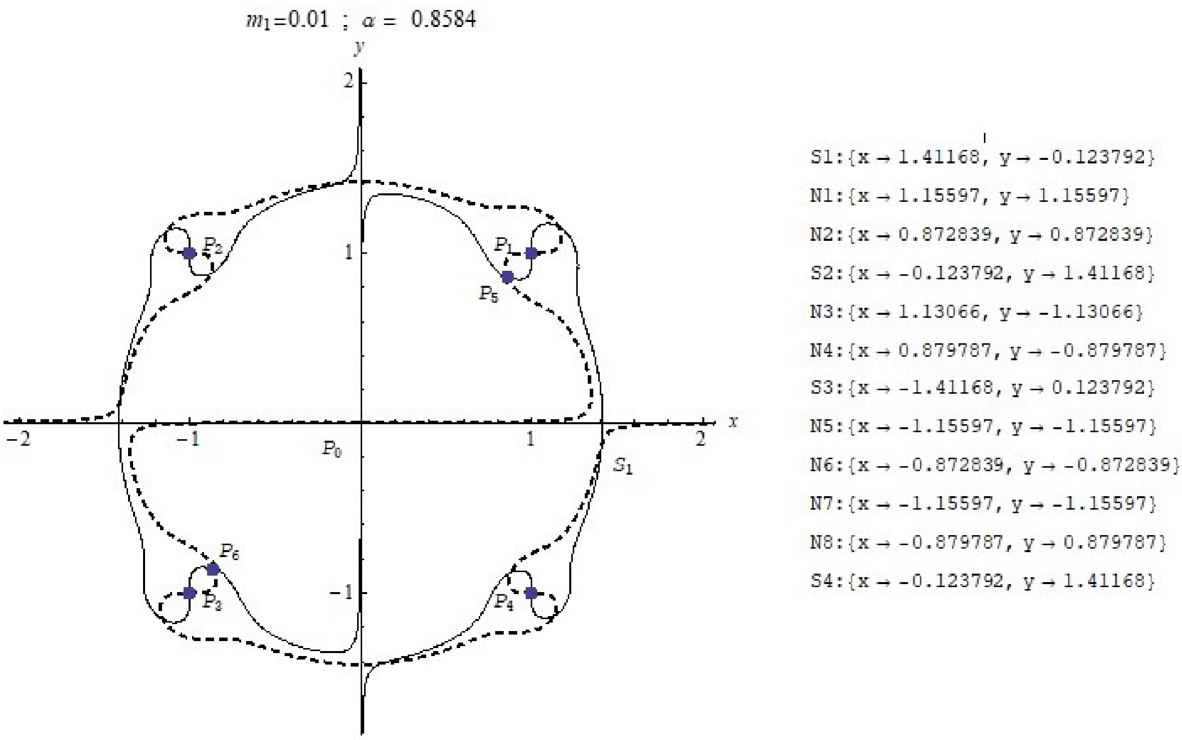

Figure 6. graph[0.01,0.8584]

The linear stability of this stationary points is studied.

Determination of the admissible variations intervals for the parameters

To study the stability of the points N_i, S_i by the first method of A.M. Lyapunov it is necessary to linearize the system of the differential equation (11) in the neighborhood of each stationary point N_i, S_i. In advance, the equations of motion of the point that passively gravitates must be written in the normal Cauchy form.

Table 2 below contains the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ for stationary points N_1 and S_1. Analyzing the Table 2 we notice that for the stationary point N_1, varying the values of

Table 2:

m_1	α	N_1		S_1	
		$\lambda_1,\ \lambda_2$	$\lambda_3,\ \lambda_4$	$\lambda_1,\ \lambda_2$	$\lambda_3,\ \lambda_4$
0.01	0.8583	± 1.30918	$\pm 1.12374\,i$	$\pm 0.28434\,i$	$\pm 0.51826\,i$
0.01	0.8584	± 1.30792	$\pm 1.12295\,i$	$\pm 0.49471\,i$	$\pm 0.32201\,i$
0.01	0.8585	± 1.30666	$\pm 1.12216\,i$	$\pm 0.45941\,i$	$\pm 0.36935\,i$
0.01	0.85853	± 1.30627	$\pm 1.12197\,i$	$\pm 0.00440 + 0.36926\,i$	$\pm 0.00440 - 0.36926\,i$
0.1	0.715	± 1.19131	$\pm 1.06789\,i$	$\pm 0.34443 + 0.53193\,i$	$\pm 0.34443 - 0.53193\,i$
0.1	0.717	± 1.17894	$\pm 1.06051\,i$	$\pm 0.40784 + 0.56449\,i$	$\pm 0.40784 - 0.56449\,i$
1	0.48965	± 1.36716	$\pm 1.30616\,i$	$\pm 0.74472 + 0.82809\,i$	$\pm 0.74472 - 0.82809\,i$
1	0.505	± 1.23329	$\pm 1.12811\,i$	$\pm 0.75807 + 0.83104\,i$	$\pm 0.75807 - 0.83104\,i$
10	0.291	± 2.50383	$\pm 2.63038\,i$	$\pm 1.6617 + 1.88497\,i$	$\pm 1.6617 - 1.88497\,i$
100	0.2	± 8.22619	$\pm 8.56881\,i$	± 15.3124	$\pm 8.390991\,i$
1000	0.2	± 27.1564	$\pm 28.0709\,i$	$\pm 17.7615 + 19.8928\,i$	$\pm 17.7615 - 19.8928\,i$

the parameters m_1 and α the eigenvalues are not purely imaginary. The same result is obtained for the other N_i points. It follows that stationary N_i points are unstable in the first approximation. We will formulate this result by the theorem:

Theorem 2. *The radial equilibrium points N_i of the differential equations describing the restricted eight body problem are unstable in the first approximation for any values of the parameters m_1 and α.*

From Table 3 we can see that in the equilibrium point S_1 for certain values of the parameters m_1 and α the eigenvalues of the matrix A are purely imaginary. Hence this stationary point S_1 is stable in the first approximation. Similarly, similar results are obtained for other points of type S_i.

Theorem 3. *There are values of the parameters m_1 and α for which the bisectorial stationary points S_i of the restricted eight body problem are stable in the first approximation.*

Moreover, we obtain that only for $0.85812 < \alpha < 0.85854$ and $m_1 = 0.01$ there are stationary points in the research problem that are linearly stable.

Concluding remarks

We have determined sufficient existence conditions of configuration describing the restricted Newtonian eight bodies problem. We have used some built in functions of the Mathematica programming environment in order to determine the stationary points. Their linear stability has been studied. It has been demonstrated that there are values of the parameters m_1 and α for which the bisectorial stationary points are stable in the first approximation. Intervals of stability and instability for geometric parameter are found.

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