Exciton–Polariton Parametric Oscillator Dynamics in the Semiconductor Microcavity

O. F. VASILIEVA

T.G. Shevchenko Pridnestrovian State University, 25 October 128, Tiraspol, 3300 MD Moldova
e-mail: florina_of@mail.ru

***Abstract* — The dynamics of an exciton–polariton parametric oscillator has been investigated. A system of nonlinear differential equations has been obtained for describing the time evolution of the densities of pump and signal polaritons. It has been demonstrated that, depending on the system parameters, the time evolution of the density of polaritons can occur in the periodic and aperiodic regimes, as well as at rest. The possibility of exercising phase control of the time evolution of the system has been proved.**

***Index Terms* – exciton–polariton; microcavity; parametric scattering; pump, signal, idler modes .**

Mixed exciton–photon states in planar semiconductor microcavities with quantum wells in the active layer form a new class of quasi two dimensional particles with unique properties [1–7]. These states are referred to as the microcavity exciton polaritons. They arise as a result of the strong coupling between excitons and normal modes of electromagnetic wave radiation from semiconductor microcavities. In the strong coupling regime, the exciton and photon modes are pushed apart and, consequently, the upper and lower microcavity polariton modes appear. The photon component of the polariton determines its small effective mass, whereas the exciton component of the polariton is responsible for the effective polariton–polariton interactions, owing to which the polaritons can be scattered from one another. The nonparabolicity of the lower polariton branch allows for the occurrence of a parametric process. As a result of this process, two pump polaritons are scattered into the signal and idler modes with the energy and momentum conservation. In this respect, great interest has been expressed by researchers in the processes of polariton–polariton scattering, owing to which the exciton–polariton system exhibits strongly nonlinear properties [6–7].

It should be noted that, up to now, there have been no works in which the specific features of the dynamics of the system of polaritons in a microcavity would be considered comprehensively. Therefore, further investigation in this field is an important problem.

The purpose of the present work is to investigate the dynamics of exciton polaritons in the parametric oscillator mode.

We consider the simplest model based on the assumption that the process of exciton–polariton parametric oscillation may be described by only three quantum states, such as the pump, signal and idler states. Under the assumption that the dynamics takes place in the lower polariton branch and the population of the upper polariton branch remains negligible, the description can be simplified by restricting it to the lower polariton only. It was shown in [4, 5, 7] that upon excitation of exciton–polaritons on the lower branch of the dispersion law the process of parametric scattering of two pump polaritons (p) into polaritons of signal (s) and idler (i) modes and vise versa is described by the Hamiltonian of the following form

$$H = \hbar\omega_p \hat{a}_p^+ \hat{a}_p + \hbar\omega_s \hat{a}_s^+ \hat{a}_s + \hbar\omega_i \hat{a}_i^+ \hat{a}_i + \\ + \hbar\mu\left(\hat{a}_p^+ \hat{a}_p^+ \hat{a}_s \hat{a}_i + \hat{a}_s^+ \hat{a}_i^+ \hat{a}_p \hat{a}_p\right), \quad (1)$$

where ω_p, ω_s and ω_i are the frequencies of the pump, signal and idler modes respectively, μ is the constant of parametric process, $\hat{a}_p$, $\hat{a}_s$ and $\hat{a}_i$ are the annihilation Bose–operators of polaritons of corresponding modes. The first term in brackets describes scattering from the pump into the signal and idler, and the last one accounts for the reverse process. We postulate here the existence of coherent coupling between pump, signal and idler modes. In this case their phases may be synchronized, wich is achieved by the coherent transfer of polaritons between three modes. The detailed analysis of the blueshifts due to the interparticle interactions lies beyond the scope of the present paper, because we consider the peculiarities of the simplest, parametric interaction between two pump polaritons and a pair of signal and idler polaritons.

Using (1), we can obtain a system of Heisenberg equations for these operators. Averaging this system of equations and applying the mean field approximation yield a system of nonlinear evolution equations for the complex amplitudes $a_{p,s,i} = \langle \hat{a}_{p,s,i} \rangle$ in the form

$$i\dot{a}_p = (\omega_p - i\gamma_p)a_p + 2\mu a_p^* a_s a_i,$$
$$i\dot{a}_s = (\omega_s - i\gamma_s)a_s + \mu a_i^* a_p a_p,$$
$$i\dot{a}_i = (\omega_i - i\gamma_i)a_s + \mu a_s^* a_p a_p, \quad (2)$$

where γ_p, γ_s and γ_i are the phenomenological damping constants of the respective mode, which are due to leaving of polaritons from the coherent states. The system of equations (2) should be supplemented by the initial conditions, which can be written in the form

$$a_{p|t=0} = a_{p0}\exp(i\varphi_{p0}), \ a_{s|t=0} = a_{s0}\exp(i\varphi_{s0}),$$
$$a_{i|t=0} = a_{i0}\exp(i\varphi_{i0}), \quad (3)$$

where $a_{p0} = \sqrt{N_{p0}}$, $a_{s0} = \sqrt{N_{s0}}$, $a_{i0} = \sqrt{N_{i0}}$ and φ_{p0}, φ_{s0}, φ_{i0} are respectively the real amplitudes and phases of polaritons at the initial moment of time. Further we will introduce the polariton densities $N_p = a_p^* a_p$, $N_s = a_s^* a_s$, $N_i = a_i^* a_i$ and two functions $Q = i(a_p a_p a_s^* a_i^* - a_s a_i a_p^* a_p^*)$ and $R = a_p a_p a_s^* a_i^* + a_s a_i a_p^* a_p^*$. Using (2) it is easy to obtain a new system of nonlinear differential equations for these functions:

$$\dot{N}_p = -2\gamma_p N_p + 2\mu Q,\ \dot{N}_s = -2\gamma_s N_s - \mu Q,$$
$$\dot{N}_i = -2\gamma_i N_i - \mu Q,\ \dot{Q} = \Delta R - (2\gamma_p + \gamma_s + \gamma_i)Q + $$
$$+ 2\mu\left(4N_p N_s N_i - N_p^2 N_s - N_p^2 N_i\right),$$
$$\dot{R} = -\Delta Q - (2\gamma_p + \gamma_s + \gamma_i)R, \qquad (4)$$

where $\Delta = 2\omega_p - \omega_s - \omega_i$.

is the resonance detuning. The initial conditions for the system of equations (4) can be written as follows: $Q_{|t=0} = 2N_{p0}\sqrt{N_{s0}N_{i0}}\sin\theta_0$, $N_{p|t=0} = N_{p0}$, $N_{s|t=0} = N_{s0}$, $N_{i|t=0} = N_{i0}$, $R_{|t=0} = 2N_{p0}\sqrt{N_{s0}N_{i0}}\cos\theta_0$, where $\theta_0 = \varphi_{s0} + \varphi_{i0} - 2\varphi_{p0}$ is the initial phase difference. We suppose that the initial conditions we can produce by the action of supershort (δ-shaped) pulses of laser radiation.

It follows from equation (4) that there is no possibilities to obtain exact analytical solutions of this system. That is why we consider the two limit cases. One of them is the limit γ_p, γ_s, $\gamma_i \to 0$. This is the case of the evolution at the times which are small in comparison with the exciton–polariton relaxation times. In this case the process of relaxation will not come into action and the evolution of the system represents the limit of the optical exciton–polariton nutation after the action of supershort pulse of laser radiation, which is responsible for the generation of the initial state of the system. Assuming, that $\gamma_p = \gamma_s = \gamma_i = 0$, we obtain the conservative system of equations:

$$\dot{N}_p = 2\mu Q,\ \dot{N}_s = -\mu Q,\ \dot{N}_i = -\mu Q,\ \dot{R} = -\Delta Q$$
$$\dot{Q} = \Delta R + 2\mu\left(4N_p N_s N_i - N_p^2 N_s - N_p^2 N_i\right), \qquad (5)$$

We consider in detail the time evolution of the system for the case $\gamma_p = \gamma_s = \gamma_i = 0$, i.e. in the absence of damping. Using the system of equations (5), we obtain the following integrals of motion:

$$N_p + 2N_s = N_{p0} + 2N_{s0},\ N_p + 2N_i = N_{p0} + 2N_{i0},$$
$$Q^2 + R^2 = 4N_p^2 N_s N_i,\ R = R_0 + \frac{\Delta}{2\mu}\left(N_{p0} - N_p\right). \qquad (6)$$

It follows from (6) that nontrivial evolution of the system can take place in the case, when if only two of the initial densities of particles are nonzero. This is due to the complicated stimulation of the process of four-wave interaction, since we take into account only the stimulated transitions.

It is more comfortable to carry out the further consideration for the normalized values of the density of pump polaritons $y = N_p / N_{p0}$. Then the system of equations (5) we can represent as a single nonlinear differential equation

$$\frac{1}{2}\left(\frac{dy}{d\tau}\right)^2 + W(y) = 0, \qquad (7)$$

where

$$W(y) = W_1(y) + W_2(y),$$
$$W_1(y) = -2y^2\left(1 + 2\overline{N}_{s0} - y\right)\left(1 + 2\overline{N}_{i0} - y\right),$$
$$W_2(y) = 2\left(2\sqrt{\overline{N}_{s0}\overline{N}_{i0}}\cos\theta_0 + \alpha(1 - y)\right)^2. \qquad (8)$$

Here $\overline{N}_{s0} = N_{s0}/N_{p0}$, $\overline{N}_{i0} = N_{i0}/N_{p0}$, $t = \tau\tau_0$, $\tau_0^{-1} = \mu N_{p0}$, $\alpha = \Delta/(2\mu N_{p0})$.

The equation (7) describes the evolution of the nonlinear oscillator, where $W(y)$ plays the role of the potential energy of this oscillator and $\frac{1}{2}(dy/d\tau)^2$ is the kinetic energy. Now it is easy to determine the qualitative behavior of the function $y(\tau)$, investigating the dependence of the potential energy $W(y)$ on y for the different values of the parameters. The evident form of function $y(\tau)$ is determined by the roots of the algebraic equation $W(y) = 0$, which depend on the parameters $\overline{N}_{s0}$, $\overline{N}_{i0}$, θ_0, α, τ_0.

We will consider at first the time evolution of the system for the initial phase difference $\theta_0 = \pi/2$. Then the equation $W(y) = 0$ has four real roots, which we arrange in the order of decrease of their values and designate them $y_1 > y_M > y_m > y_4$ respectively. In the limit of small values of α they change with respect to α:

$$y_1 = 1 + 2\overline{N}_{s0} + \frac{2\alpha^2\overline{N}_{s0}^2}{\left(\overline{N}_{s0} - \overline{N}_{i0}\right)\left(1 + 2\overline{N}_{s0}\right)^2},$$
$$y_M = 1 + 2\overline{N}_{i0} - \frac{2\alpha^2\overline{N}_{i0}^2}{\left(\overline{N}_{s0} - \overline{N}_{i0}\right)\left(2\overline{N}_{i0} + 1\right)^2},$$
$$y_m = \frac{|\alpha|}{\sqrt{\left(1 + 2\overline{N}_{s0}\right)\left(1 + 2\overline{N}_{i0}\right)}},\ y_4 = -y_m.$$

Here the roots y_M and y_m have the meaning of maximal and minimal normalized densities of pump polaritons, which they can have during the evolution process. Further for the definiteness we will suggest that $\overline{N}_{i0} > \overline{N}_{s0}$. Evolution of roots depending on parameter α is presented in Fig.1. We can see that the roots y_1 and y_m increase, but y_M and y_4 decrease with the increase of α. Then the solution of the equation (10) has the form:

$$y = \frac{y_M(y_1 - y_m) - y_1(y_M - y_m)\mathrm{sn}^2(x)}{y_1 - y_m - (y_M - y_m)\mathrm{sn}^2(x)},$$
$$x = \sqrt{(y_1 - y_m)(y_M - y_4)}\,\tau \pm f(\varphi_0, k) \qquad (9)$$

where $\mathrm{sn}(x)$ is the elliptic sin, $f(\varphi_0, k) = F(\varphi_0, k) - K(k)$, $F(\varphi_0, k)$ is the incomplete elliptic integral of the first kind with the modulus k and parameter φ_0, $K(k)$ is the complete elliptic integral of the first kind. The quantities k and φ_0 are determined by the expressions:

$$k^2 = \frac{(y_1 - y_4)(y_M - y_m)}{(y_1 - y_m)(y_M - y_4)}, \varphi_0 = \arcsin\sqrt{\frac{(y_M - y_4)(1 - y_m)}{(y_M - y_m)(1 - y_4)}}.$$

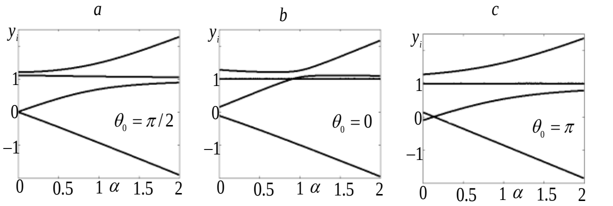

Fig.1. Dependence of the roots y_i ($i = 1,2,3,4$) of the equation $W(y) = 0$ on the parameter α for $\overline{N}_{s0} = 0.1$, $\overline{N}_{i0} = 0.05$ and different values of θ_0, equals to $\pi/2$, 0 and π.

From (13) we can obtain the amplitude A and period T of oscillations of the density of pump polaritons:

$$A = y_M - y_m,\ T = 2K(k)\big/\sqrt{(y_1 - y_m)(y_M - y_4)} \quad (10)$$

Here the amplitude A is determined as the difference between the second and the third roots of the equation $W(y) = 0$ (Fig. 1a).

The periodic evolution of the density of pump polaritons is presented in Fig. 2a. The polariton density changes periodically in time between the values y_m and y_M. Hence there is no total conversion of the pump polaritons into the signal and idler polaritons, i.e. the density oscillations take place without depletion of pump polaritons. The amplitude and period of the oscillations monotonously decrease with the increase of α for the fixed values of $\overline{N}_{s0}$ and $\overline{N}_{i0}$ (Fig. 2 *b*,*c*). We point out that in the absence of idler ($\overline{N}_{i0} = 0$) or signal ($\overline{N}_{s0} = 0$) polaritons at the initial moment the root y_M of the equation $W(y) = 0$ is equal to one for any α. Therefore the solution (9) is also true for the case $\overline{N}_{i0} = 0$ or $\overline{N}_{s0} = 0$ taking into account, that $y_M = 1$.

Next we will consider the evolution of the system for the phase difference $\theta_0 = 0$. From (6) we can see, that in this case one of the roots of the equation $W(y) = 0$ coincides with the initial condition $y = y_0 = 1$ (Fig. 1b). Therefore the solution will not include the phase shift. In the case, when the expression

$$4\overline{N}_{s0}\overline{N}_{i0} + 2\alpha\sqrt{\overline{N}_{s0}\overline{N}_{i0}} = \overline{N}_{s0} + \overline{N}_{i0} \quad (11)$$

is satisfied the second root equals one too (Fig. 1b).

Moreover, the solution of equation (7) in this case is $y(\tau) = y_0 = 1$, i.e. it coincides with the initial condition too, which is due to the crossing of two middle roots

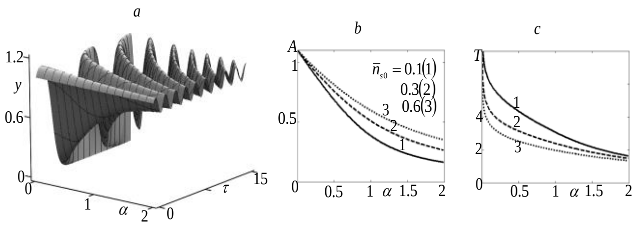

Fig.2. a) Time evolution of the normalized density of pump polaritons for $\overline{N}_{s0} = 0.1$, $\overline{N}_{i0} = 0.05$ and different values of α and the dependence b) of the amplitude A and c) of the period T of the oscillations of the density of pump polaritons on the value of $\overline{N}_{s0}$ for $\theta_0 = \pi/2$ and fixed value of $\overline{N}_{i0} = 0.05$.

depending on α. This means that it is impossible nontrivial evolution of the system in the case of nonzero initial densities of all polaritons, when the expression (18) is satisfied. On the phase plane $(y, \dot{y})$ this solution corresponds to the phase center. System is at rest if $\theta_0 = 0$. In the plot of the potential energy $W(y)$ of the nonlinear oscillator depending on y the rest corresponds to the phase point of particle, which locates in the minimum of the potential energy without kinetic energy. The particle in this case does not move from the initial state, since its velocity is equal to zero.

Two different cases of the evolution are possible, if the expression (11) is not true. The equation $W(y) = 0$ has four real roots. Depending on the relation between the parameters $\overline{N}_{s0}$, $\overline{N}_{i0}$ and α in the first case the roots are arranged in the order $y_1 > y_0 = 1 > y_m > y_4$, and in the second case $y_1 > y_M > y_0 = 1 > y_4$ (Fig. 1b). In the first case the density of pump polaritons changes within the limits $y_m \le y \le y_0 = 1$, while in the second case within the limits $y_0 = 1 \le y \le y_M$. Therefore depending on the values of parameters the oscillations of the density of pump polaritons are possible under the background with the amplitude $A = 1 - y_m$ in the first case and $A = y_M - 1$ in the second one, where the background density is equal to the initial density of the pump polaritons $y_0 = 1$. We point out that the aperiodic regime of evolution is absent for $\alpha = 0$ if $\theta_0 = 0$.

The solution of equation (10) for the first case has the form

$$y = \frac{y_1 - y_m - y_1(1 - y_m)\mathrm{sn}^2\sqrt{(y_1 - y_m)(1 - y_4)}\tau}{y_1 - y_m - (1 - y_m)\mathrm{sn}^2\sqrt{(y_1 - y_m)(1 - y_4)}\tau}, \quad (12)$$

where modulus k of the elliptic function, and amplitude A and period T of oscillations are equal to

$$k^2 = \frac{(y_1 - y_4)(1 - y_m)}{(y_1 - y_m)(1 - y_4)},\ A = 1 - y_m,$$

$$T = 2K(k)/\sqrt{(y_1 - y_m)(1 - y_4)}. \quad (13)$$

For the second case we obtain

$$y = \frac{y_M - y_4 - y_4(y_M - 1)\mathrm{sn}^2\sqrt{(y_1 - 1)(y_M - y_4)}\tau}{y_M - y_4 - (y_M - 1)\mathrm{sn}^2\sqrt{(y_1 - 1)(y_M - y_4)}\tau}, \quad (14)$$

where

$$k^2 = \frac{(y_1 - y_4)(y_M - 1)}{(y_1 - 1)(y_M - y_4)}, \quad A = y_M - 1,$$

$$T = 2K(k)/\sqrt{(y_1 - 1)(y_M - y_4)}. \quad (15)$$

If we put $y_m = 1$ in (12) or $y_M = 1$ in (14), we obtain the solution $y(\tau) = 1 = const$ again.

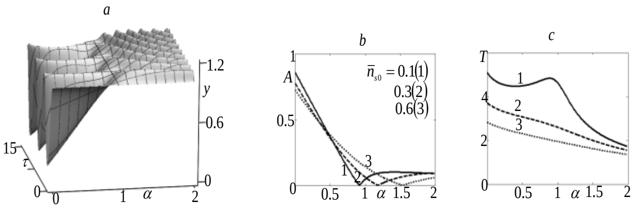

Fig.3. The same as in Fig.2, but for $\theta_0 = 0$.

It follows from (12) and (14), that the density of pump polaritons evolves periodically (Fig. 3a). As we pointed out, the oscillations of the density take place both under background (for small values of α) and above background (for great α). The amplitude of oscillations, which is defined as the difference of two middle roots of the equation $W(y) = 0$, for fixed $\bar{N}_{s0}$ and $\bar{N}_{i0}$ at first decreases with the increase of α, tends to zero at $\alpha = \alpha_c \equiv (\bar{N}_{i0} + \bar{N}_{s0} - 4\bar{N}_{i0}\bar{N}_{s0}) / (2\sqrt{\bar{N}_{i0}\bar{N}_{s0}})$, when two middle roots (y_m and $y_0 = 1$) coincides, and then increases and goes rapidly to the saturation (Fig. 3b). Period of oscillation T depends essentially on the parameters $\bar{N}_{s0}$, $\bar{N}_{i0}$ (Fig. 3c). For small $\bar{N}_{s0}$ period T at first increases, reaches maximal value at $\alpha = \alpha_c$, then decreases rapidly. At the great value of $\bar{N}_{s0}$ the period of oscillations decreases monotonously with the increase of α (Fig. 3c).

If the initial phase difference $\theta_0 = \pi$, then the roots of the equation $W(y) = 0$ for $\alpha \geq 0$ as before are arranged in an order $y_1 > y_M > y_m > y_4$ (Fig. 1c), where y_M and y_m as before play role of the maximal and minimal density of pump polaritons. It follows from the Fig. 1c, that at $\theta_0 = \pi$ a crossing between the roots y_4 and y_m appears with the increase of α. The degeneration of these roots brings about to the appearance of the aperiodic evolution of the system similar to the case when $\theta_0 = \pi/2$. Hence at $\theta_0 = \pi$ the periodic and aperiodic regimes of evolution are possible. In the case when all four roots are different the evolution of the system is described by the equation (9) and hence the density of pump polaritons changes periodically between the values y_m and y_M (Fig. 4a).

It follows from Fig. 4a that the periodic regime of evolution transits into the aperiodic one and vice versa.

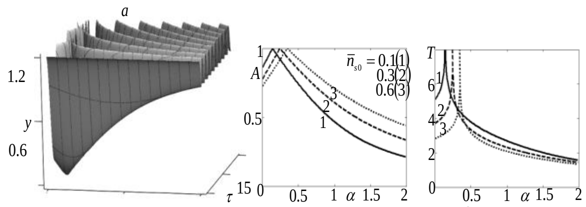

Fig.6. The same as in Fig.2, but for $\theta_0 = \pi$.

As for the amplitude A and period T of the oscillations of the pump polariton density, we can see from Figs. 4b and 4c, that with the increase of α the amplitude of oscillation at first increases, reaches one and then monotonously decreases, while the period of oscillations at first increases too with the increase of α, but then it diverges at the value α, for which both least roots (y_4 and y_1) occur to be equal, and then monotonously decreases.

In conclusion, we point out that the dynamics of polaritons in the regime of parametric oscillator represents the periodic conversion of pairs of pump polaritons into the polaritons of the signal and idler modes and vice versa. The period and amplitude of oscillations of the polariton density depend on the initial densities of polaritons, initial phase difference and resonance detuning. For the definite relation between the parameters it is possible the aperiodic evolution of the system too, which represents the conversion only the part of pump polaritons into the signal and idler polaritons, by which the evolution of the system is finished. The significant dependence of the period and amplitude of oscillations of polaritons on the initial phase difference evidences about the possibility of the phase controlling of the polariton dynamics. The similar effect was predicted earlier for the process of atomic–molecular conversion in a Bose–Einstein condensate.

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