DISCONTINUOUS SOLUTIONS IN BEM FOR PLATE ANALYSIS IN REISSNER-MINDLIN THEORY

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ABSTRACT

We present a new direction in the indirect Boundary Element Method (BEM), based on discontinuous solutions. These solutions were obtained by Prof. Moraru Gheorghe by applying the generalized Fourier transform to the differential equation of plates. Using these solutions as Green Influence functions we can solve plate bending problems that other analytical and numerical methods have no solution or have solving difficulties, for example: plates of arbitrary shapes and types of support, different loads, the presence of defects etc. For the proposed method, we performed a numerical implementation of the discontinuous solutions and developed a computational program in the Matlab programming language. In order to highlight the effect of transverse shear deformations on the deflection in Reissner-Mindlin plate theory, using this program, we calculated the displacements and stresses in square plates for different ratios of thickness to side length. The obtained results were compared with the Finite Element Method (FEM) and with analytical solutions (Fourier trigonometric series) in the classical plate theory.

Keywords: Boundary Element Method; discontinuous solution; Green function; Reissner; Mindlin; plate.

INTRODUCTION

One of the most used numerical method in different areas (mechanics of deformable body, the mechanic of a liquid, magnetic fields etc.) is the finite element method (FEM) [6]. This method possesses a number of advantages in comparison with other methods: flexibility, easy for programming, is effective for nonlinear problems etc. At the same time FEM has some disadvantages. For example: a massive system of linear equations, necessity of big initial data, requires big volume of computer memory, difficulties in solving problems that present: stress concentrations, defects, contact problems, connecting plate (shell) elements with bars etc.

Recently the boundary elements method (BEM) intensively develops [1-3]. In BEM the discretization is applied only to the edge of the plate that reduce the number of nodes and elements, also reducing the size of the system of linear equations by one unit. The solutions inside the domain are continuous and provide more accurate results.

In the theory of plates and shells there are two approaches to obtain the integral equations: direct and indirect. The direct method [2, 12] which is based on fundamental solutions does not allow satisfying every possible boundary condition on the edge.

For the indirect BEM [10, 16] is offered a new approach which is based on discontinuous solutions. These solutions give the possibility to formulate the integral equations or systems of integral equations for various cases of plates and shells. The method allows to consider the behavior of the solutions in singular points and it is suitable for problems of the theory of plates and shells with defects (cracks, elastic or rigid inclusions etc.).

We present the discontinuous solutions for plates in Reissner-Mindlin plate theory [5-11]. In classical theory of plates [15], the fourth order differential equation for deflection of the mid-plane allows us to satisfy only two conditions along each edge instead of three. The impossibility of satisfying more than two conditions is caused by the neglect of the transverse deformations. And this leads to the appearance of reactions concentrated in the corners of the plates. For thin plates, neglecting the transverse shear deformations practically does not affect the results, but for thick plates, as the ratio of thickness to side length increases, this can have a considerable influence.

MATERIALS AND METHODS

1. Governing equations in Reissner-Mindlin plate theory.

Consider an infinite plate of thickness *h*. According to Reissner-Mindlin plate theory the deflection *w* of the mid-plane is governed by a system of two differential equations.

$$\left.\begin{aligned} D\Delta\Delta w &= q - \frac{h^2(2-\nu)}{10(1-\nu)}\Delta q; \\ \Delta\psi &- \frac{10}{h^2}\psi = 0, \end{aligned}\right\} \quad (1)$$

where Δ is the Laplacian, q is the transverse load per unit area; $D = Eh^3/12\left(1-\nu^2\right)$ s the flexural rigidity, E and ν are Young's modulus and Poisson's ratio, respectively; ψ is the stress function.

The first differential equation of the system allows to satisfy only two conditions for each side. The second one represents the supplementary equation that offers the possibility to satisfy one more condition.

2. The solutions due to the concentrated jumps.

Let us suppose that in the infinite plate on the axis y ($x = 0$) (fig. 1) there is a defect (crack, plastic hinge, inclusion etc.). When passing from one side of the defect $x=-0$ the other $x=+0$ e displacement w, the slope angles θ_x and θ_y, the bending moment M_x, the twisting moment M_{xy} and the shear force Q_x could have jumps. For these jumps we introduce the following notation:

$$\begin{aligned} &w(-0,y)-w(+0,y)=\langle w(y)\rangle; \\ &\theta_x(-0,y)-\theta_x(+0,y)=\langle \theta_x(y)\rangle; \\ &\theta_y(-0,y)-\theta_y(+0,y)=\langle \theta_y(y)\rangle; \\ &M_x(-0,y)-M_x(+0,y)=\langle M_x(y)\rangle; \\ &M_{xy}(-0,y)-M_{xy}(+0,y)=\langle M_{xy}(y)\rangle; \\ &Q_x(-0,y)-Q_x(+0,y)=\langle Q_x(y)\rangle. \end{aligned} \quad (2)$$

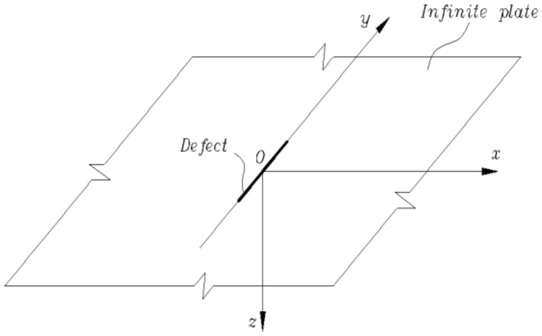


Fig. 1. Infinite plate with a defect.

The solutions due to the concentrated jumps we obtain by applying the generalized Fourier transform [14] to equations (1) on supposing that $q(x,y) = 0$.

The relation between concentrated jumps and displacements can be written in the following form:

$$\begin{Bmatrix} w(x,y) \\ \theta_x(x,y) \\ \theta_y(x,y) \end{Bmatrix} = \begin{bmatrix} g_{11} & g_{12} & g_{13} & g_{14} & g_{15} & g_{16} \\ g_{21} & g_{22} & g_{23} & g_{24} & g_{25} & g_{26} \\ g_{31} & g_{32} & g_{33} & g_{34} & g_{35} & g_{36} \end{bmatrix} \begin{Bmatrix} \langle w(y) \rangle \\ \langle \theta_x(y) \rangle \\ \langle \theta_y(y) \rangle \\ \langle M_x(y) \rangle \\ \langle M_{xy}(y) \rangle \\ \langle Q_x(y) \rangle \end{Bmatrix}, \qquad (6)$$

the relations between jumps and efforts are given by:

$$\begin{Bmatrix} M_x(x,y) \\ M_y(x,y) \\ M_{xy}(x,y) \\ Q_x(x,y) \\ Q_y(x,y) \end{Bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} & t_{14} & t_{15} & t_{16} \\ t_{21} & t_{22} & t_{23} & t_{24} & t_{25} & t_{26} \\ t_{31} & t_{32} & t_{33} & t_{34} & t_{35} & t_{36} \\ t_{41} & t_{42} & t_{43} & t_{44} & t_{45} & t_{46} \\ t_{51} & t_{52} & t_{53} & t_{54} & t_{55} & t_{56} \end{bmatrix} \begin{Bmatrix} \langle w(y) \rangle \\ \langle \theta_x(y) \rangle \\ \langle \theta_y(y) \rangle \\ \langle M_x(y) \rangle \\ \langle M_{xy}(y) \rangle \\ \langle Q_x(y) \rangle \end{Bmatrix}, \qquad (7)$$

where elements g_{ij} and t_{ij} are given:

$$g_{11} = -\frac{1}{2\pi}\frac{x}{r^2};\ \ldots;\quad t_{11} = \frac{D(1-\nu)}{\pi}\left[\frac{x^3-3xy^2}{r^6} - \frac{1}{2}\frac{\partial^3}{\partial x \partial y^2}K_0(\lambda r)\right];\ \ldots \qquad (8)$$

3. The efforts in plates with arbitrary boundary

Using the solutions due to the concentrated jumps as Green functions [4, 13] by superposition we can obtain the discontinuous solutions for the defect placed on the contour *L* (fig. 2).

Using the coordinate transformation from the local system of coordinates $(\bar{x}, \bar{y})$ o the local system (*n*,*t*) we can write:

$$\begin{aligned}
w^*(P) &= \int_L \bar{w}(P,Q)\,ds_Q; \\
\theta_n^*(P) &= \int_L \left[\bar{\theta}_x(P,Q)\cos\gamma + \bar{\theta}_y(P,Q)\sin\gamma\right] ds_Q; \\
\theta_t^*(P) &= \int_L \left[-\bar{\theta}_x(P,Q)\sin\gamma + \bar{\theta}_y(P,Q)\cos\gamma\right] ds_Q; \\
M_n^*(P) &= \int_L \left[\bar{M}_x(P,Q)\cos^2\gamma + \bar{M}_y(P,Q)\sin^2\gamma + 2\bar{M}_{xy}(P,Q)\cos\gamma\sin\gamma\right] ds_Q; \\
M_{nt}^*(P) &= \int_L \left\{\left[\bar{M}_y(P,Q) - \bar{M}_x(P,Q)\right]\cos\gamma\sin\gamma + \bar{M}_{xy}(P,Q)\left(\cos^2\gamma - \sin^2\gamma\right)\right\} ds_Q; \\
Q_n(P) &= \int_L \left[Q_x(P,Q)\cos\gamma + Q_y(P,Q)\sin\gamma\right] ds_Q,
\end{aligned} \qquad (9)$$

where $\gamma = \beta - \alpha$

hese solutions also can be used for solving basic problems of plate bending. In these cases, the boundary will be considered as a defect into an infinite plate. By approaching the edge from the inside of the region occupied by the plate, the jumps will be considered equal to the values on the boundary. When approaching the edge from the outside these jumps will be considered zero.

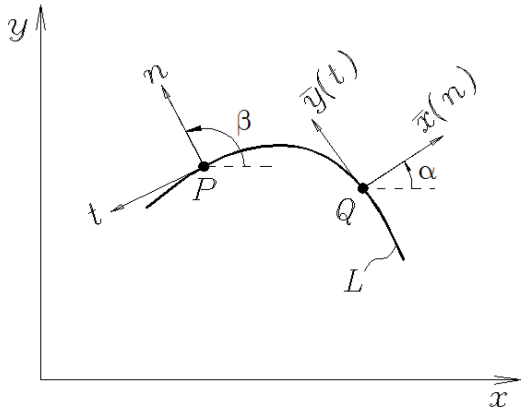


Fig. 2. Local systems of coordinates on contour *L*.

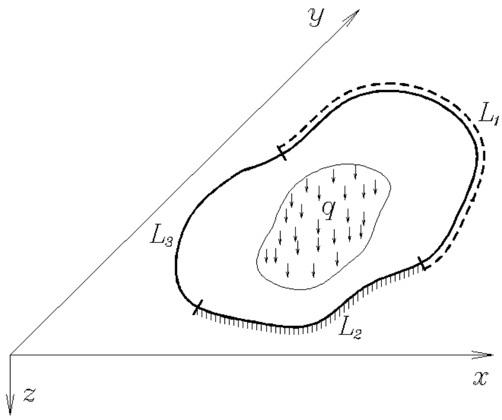


Fig. 3. Plate of arbitrary shape.

4. Numerical implementation of discontinuous solutions in BEM.

Consider a plate of arbitrary shape (fig. 3). On the edge L_1 the plate is simply supported, on L_2 – clamped and on L_3 – free.

To obtain integral equations, the deformable state of the plate is presented as a sum of two states. The first one (marked with a circle) due to the transversal loads. The second (marked with an asterisk) due to the concentrated jumps on the line *L* of the defect.

The boundary conditions are:

– for simply supported edge (L_1)

$$w^* + w^o = 0; \quad M_n^* + M_n^o = 0; \; M_{nt}^* + M_{nt}^o = 0;$$

– for clamped edge (L_2)

$$w^* + w^o = 0; \; \theta_n^* + \theta_n^o = 0; \; \theta_t^* + \theta_t^o = 0;$$

– for free edge (L_3)

$$M_n^* + M_n^o = 0; \; M_{nt}^* + M_{nt}^o = 0; \; Q_n^* + Q_n^o = 0.$$

The solutions due to transversal load depend of the type of the load. For example, if the plate is loaded by a force *F* in the point with coordinates a_0, b_0, then:

$$w_i^o = F \cdot g_{16\,i}\left(x_i^m - a_0, y_i^m - b_0\right);$$

$$\vdots$$

$$M_{ni}^o = F\left(n_x^2 M_{xi}^o + n_y^2 M_{yi}^o + 2 n_x n_y M_{xyi}^o\right) = F\left[n_x^2 t_{x\,16\,i}\left(x_i^m - a_0, y_i^m - b_0\right) + n_y^2 t_{y\,26\,i}\left(x_i^m - a_0, y_i^m - b_0\right) + 2 n_x n_y t_{xy\,36\,i}\left(x_i^m - a_0, y_i^m - b_0\right)\right], \qquad (10)$$

$$\vdots$$

where $n_x = \cos\alpha$ şi $n_y = \sin\alpha$

For other loading cases the solutions can be obtained by integrating the expressions (10).

If we discretize the contour L into a set of constant elements we obtain the following system of equations:

$$\begin{cases}
\sum_{j=n_{L3}} w^1_{ij}\langle w_j\rangle + \sum_{j=n_{L1},n_{L3}} w^2_{ij}\langle\theta_{nj}\rangle + \sum_{j=n_{L1},n_{L3}} w^3_{ij}\langle\theta_{tj}\rangle + \sum_{j=n_{L2}} w^4_{ij}\langle M_{nj}\rangle + \sum_{j=n_{L2}} w^5_{ij}\langle M_{ntj}\rangle + \sum_{j=n_{L1},n_{L2}} w^6_{ij}\langle Q_{nj}\rangle = -w^o_i; & (i=n_{L1},n_{L2}) \\
\sum_{j=n_{L3}} \theta^{n1}_{ij}\langle w_j\rangle + \sum_{j=n_{L1},n_{L3}} \theta^{n2}_{ij}\langle\theta_{nj}\rangle + \sum_{j=n_{L1},n_{L3}} \theta^{n3}_{ij}\langle\theta_{tj}\rangle + \sum_{j=n_{L2}} \theta^{n4}_{ij}\langle M_{nj}\rangle + \sum_{j=n_{L2}} \theta^{n5}_{ij}\langle M_{ntj}\rangle + \sum_{j=n_{L1},n_{L2}} \theta^{n6}_{ij}\langle Q_{nj}\rangle = -\theta^o_{ni}; & (i=n_{L2}) \\
\sum_{j=n_{L3}} \theta^{t1}_{ij}\langle w_j\rangle + \sum_{j=n_{L1},n_{L3}} \theta^{t2}_{ij}\langle\theta_{nj}\rangle + \sum_{j=n_{L1},n_{L3}} \theta^{t3}_{ij}\langle\theta_{tj}\rangle + \sum_{j=n_{L2}} \theta^{t4}_{ij}\langle M_{nj}\rangle + \sum_{j=n_{L2}} \theta^{t5}_{ij}\langle M_{ntj}\rangle + \sum_{j=n_{L1},n_{L2}} \theta^{t6}_{ij}\langle Q_{nj}\rangle = -\theta^o_{ti}; & (i=n_{L2}) \\
\sum_{j=n_{L3}} m^1_{ij}\langle w_j\rangle + \sum_{j=n_{L1},n_{L3}} m^2_{ij}\langle\theta_{nj}\rangle + \sum_{j=n_{L1},n_{L3}} m^3_{ij}\langle\theta_{tj}\rangle + \sum_{j=n_{L2}} m^4_{ij}\langle M_{nj}\rangle + \sum_{j=n_{L2}} m^5_{ij}\langle M_{ntj}\rangle + \sum_{j=n_{L1},n_{L2}} m^6_{ij}\langle Q_{nj}\rangle = -M^o_{ni}; & (i=n_{L1},n_{L3}) \\
\sum_{j=n_{L3}} h^1_{ij}\langle w_j\rangle + \sum_{j=n_{L1},n_{L3}} h^2_{ij}\langle\theta_{nj}\rangle + \sum_{j=n_{L1},n_{L3}} h^3_{ij}\langle\theta_{tj}\rangle + \sum_{j=n_{L2}} h^4_{ij}\langle M_{nj}\rangle + \sum_{j=n_{L2}} h^5_{ij}\langle M_{ntj}\rangle + \sum_{j=n_{L1},n_{L2}} h^6_{ij}\langle Q_{nj}\rangle = -M^o_{nti}; & (i=n_{L1},n_{L3}) \\
\sum_{j=n_{L3}} q^1_{ij}\langle w_j\rangle + \sum_{j=n_{L1},n_{L3}} q^2_{ij}\langle\theta_{nj}\rangle + \sum_{j=n_{L1},n_{L3}} q^3_{ij}\langle\theta_{tj}\rangle + \sum_{j=n_{L2}} q^4_{ij}\langle M_{nj}\rangle + \sum_{j=n_{L2}} q^5_{ij}\langle M_{ntj}\rangle + \sum_{j=n_{L1},n_{L2}} q^6_{ij}\langle Q_{nj}\rangle = -Q^o_{ni}. & (i=n_{L3})
\end{cases} \tag{11}$$

where the elements w^1_{ij}

, w^2_{ij}

, ... , q^6_{ij}

can be obtained by integrating the solutions from concentrated jumps on the length of the element. For example:

$$w^1_{ij} = \int_{l_j} \underline{g}_{11}\left(x^m_i, y^m_i - \eta\right) d\eta \quad \ldots; \; \theta^{n1}_{ij} = \int_{l_j}\left[\mathrm{c}\,\underline{g}_{21}\left(x^m_i, y^m_i - \eta\right) + \mathrm{s}\,\underline{g}_{31}\left(x^m_i, y^m_i - \eta\right)\right] d\eta; \; \ldots \tag{12}$$

where c = cosγ, iar s = sinγ.

By solving the system of equations (11) all the jumps on the boundary will be known, so that the displacements and the efforts in any point inside the plate can be calculated, these being expressed by the obtained jumps. For example, if it is necessary to calculate the displacement at any point K from the interior of the plate, the expression take the form:

$$w_k = \sum_{j=n_{L3}} g_{11}\langle w_j\rangle + \sum_{j=n_{L1},n_{L3},n_{Ld}} g_{12}\langle\theta_{nj}\rangle + \sum_{j=n_{L1},n_{L3}} g_{13}\langle\theta_{tj}\rangle + \sum_{j=n_{L2}} g_{14}\langle M_{nj}\rangle + \sum_{j=n_{L2}} g_{15}\langle M_{ntj}\rangle + \sum_{j=n_{L1},n_{L2}} g_{16}\langle Q_{nj}\rangle + w^o_k. \tag{13}$$

In the same maner, at any point, can be obtained the expresions for slope angles, moments and shear forces.

RESULTS

Based on the discontinuous solutions, a calculation program was developed in the Matlab programming language.

We examine a square plate, simply supported, loaded by a force F in the center (fig. 5). Using this program, we calculated the deflections (fig.6) and efforts in Reissner-Mindlin plate theory for different ratios of thickness to side length (h/a). The plate boundary has been discretized into 20 constant elements (fig. 4, a). The obtained results were compared with the FEM for a mesh 20x20 elements (fig. 4, b) and with analytical solutions (Fourier trigonometric series) in the classical plate theory.

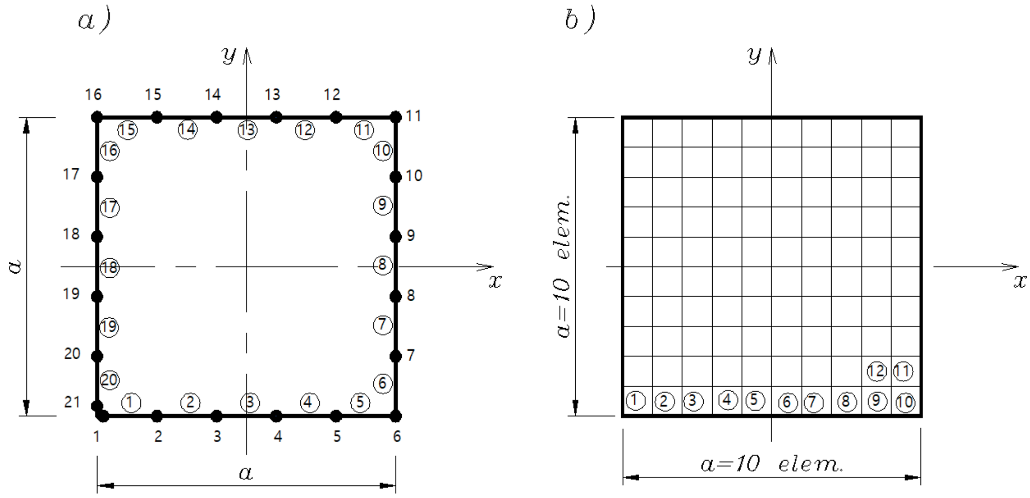
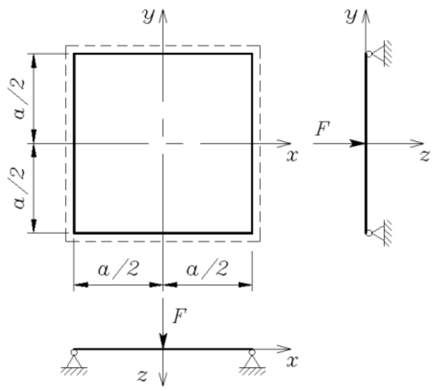


Fig. 4. Plate mesh: a) BEM; b) FEM.



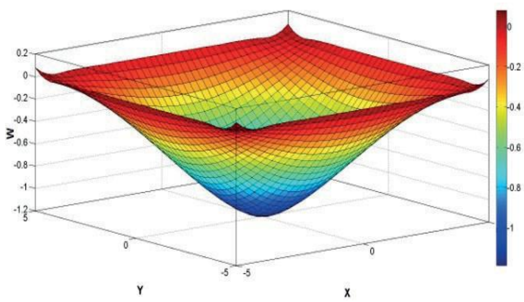
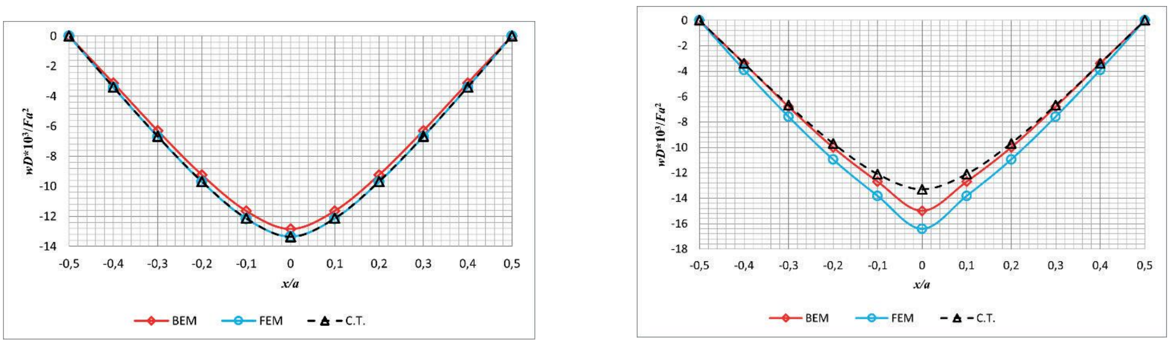


Fig. 6. Deflection field *w* for *h*/*a*=0.01.

Fig. 5. Simply supported square plate.

The results obtained on a central section ($y = 0$) using all three methods are illustrated, for comparison, in the form of diagrams (fig. 7) for different ratios h/a.

a) b)



c) d)

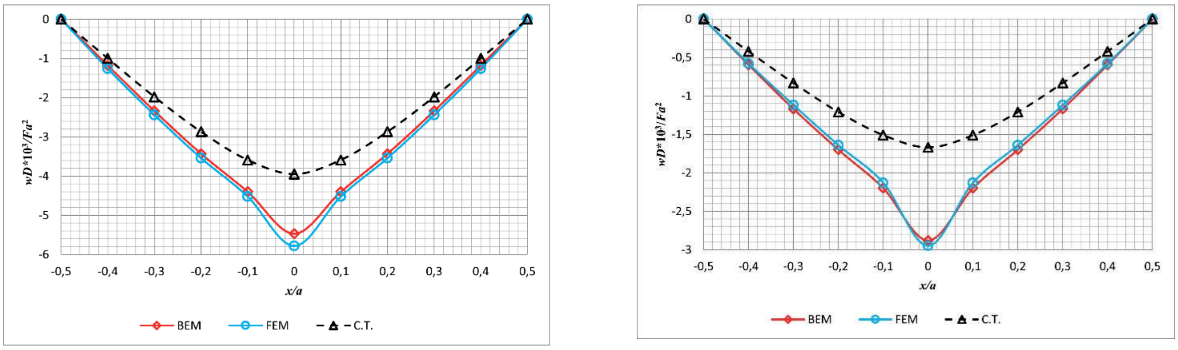


Fig. 7. Values of deflections on a central section for ratio: a) h/a = 0,01; b) h/a = 0,1; c) h/a = 0,15; d) h/a = 0,2.

These deviations as well as the values of the deflections are presented in table 1.

Table 1. Results and deviations

h/a	**Classic theory (Fourier series)**	**Reissner-Mindlin theory**			
		FEM (20x20 elem.)		**BEM (20 elem.)**	
	w	***w***	**Δ*w*, %**	***w***	**Δ*w*, %**
0,01	13,36	13,35	0,07	12,85	3,8
0,1	13,30	16,41	23,4	15,00	12,8
0,15	3,95	5,78	46,3	5,47	38,5
0,2	1,67	2,95	100,0	2,88	72,5

DISCUSSION

From the diagrams above (fig. 7) we observe that with the increase of the ratio h/a the influence of the transverse shear deformations, on the deflections becomes more and more pronounced. For thin plates with ratio $h/a < 0,01$ these shear deformations practically do not affect the deflections and the results by all three methods almost match. Shear effect is increasing near the point of application of the concentrated force. As recommended by the specialized textbooks, for ratios $h/a > 0,1$, this influence can no longer be neglected, as the deviations are considerable compared to the classical theory.

CONCLUSION

In this paper, we present the indirect boundary element method based on discontinuous solutions for plates in Reissner-Mindlin theory. The proposed method allows us to solve: plate bending problems in classical and Reissner-Mindlin theory, plates of arbitrary shapes and types of support, with different loads, the presence of defects etc. One of the main advantages of the BEM is that the discretization is applied only to the edge of the plate reducing the size of the system of linear equations by one unit, leading to a minimal use of computer resources. Also, the solutions inside the domain are continuous and provide more accurate results.

The discontinuous solutions, described in this paper, present a new direction in the field of mechanics of solids. The computing program developed on these solutions can be recommended to engineers to solve practical plate bending problems.

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