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Once again about the reliability of serial-parallel networks vs parallel-serial networks

Leahu Alexei[1], Andrievschi-Bagrin Veronica[2], Ciorbă Dumitru[3], Fiodorov Ion [4]
Technical University of Moldova, 168 Stefan cel Mare Av., MD-2004 Chisinau, Republic of Moldova.

[1] alexei.leahu@ati.utm.md, [2] veronica.bagrin@ati.utm.md,
[3] dumitru.ciorba@fcim.utm.md, [4] ion.fiodorov@ati.utm.md

***Abstract*— The paper addresses the issue of comparing the reliability of two standard types of networks: serial-parallel and parallel-serial. Four variants of dynamic mathematical models are analyzed depending on the lifetime cumulative distribution function of each units of the network, the non-random / random character of the number of units in each subnet and of the number of subnets. Sufficient conditions have been determined for serial-parallel networks to be more reliable than parallel-serial networks. The main result is that these conditions do not imply the lifetime distribution of each unit but only the probabilistic distribution of the numbers of units and subsystems of the networks.**

Keywords—lifetime distribution; reliability; power series distribution; serial-parallel/parallel-serial networks

I. INTRODUCTION

In the case of networks whose structure (topology) is a complex one, in many cases, the subnets that are part of them have a series-parallel or parallel-series structure. This means that when designing networks, not infrequently, we will need to we know, possibly, which of the mentioned sub-networks is preferable in terms of their reliability.

Based on mathematical models in various hypostases of such subnets, we aim to find the most reliable subnet.

II. SERIES-PARALLEL AND PARALLEL-SERIES NETWORKS IN VARIOUS HYPOSTASES

When in our paper we talk about the reliability of series-parallel / parallel-series networks we consider dynamic mathematical models in various situations depending on the number of units in each subnet, the number of subnets, but also the lifetime distribution of each unit of network.

The figure below shows how the two networks A and B look schematically.

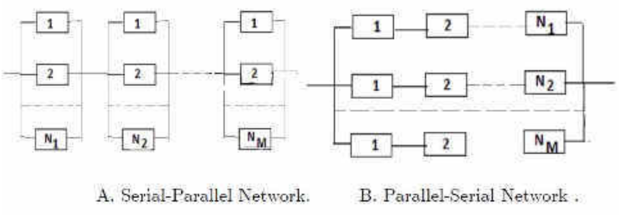

Figure 1. Schematic representation of series-parallel / parallel-series networks

We will consider, thus, that the network (regardless of its type, A or B) consists of M subnets, the subnet k consists of N_k units, $k = 1,2, ..., M,$ and the lifetimes of all units are independent, identically distributed random variables (i.i.d.r.v.) with the cumulative distribution function (c.d.f.) $F(x)$.

Even if the numbers N_k, $k = 1,2, ..., M,$ or / and M are random, we consider that they are independent of the lifetimes of all units.

Furthermore, the reliability of type A and B networks will be compared in the following variants.

Variant 1. The number of units N_k in the subnet k, $k = 1,2, ..., M$ is constant, the number of subnets being constant too.

Variant 2. The number of units N_k,, $k = 1,2, ..., M$ in the subnets $k = 1,2, ..., M$ are i.i.d.r.v. with 0-truncated Power Series Distribution (PSD) but the number of subnets being constant.

Variant 3. The number N_k of units in the subnet k, $k = 1,2, ..., M$ is constant, in addition, $N_1 = N_2 = ... = N_M = N,$ and the number M of the subnets is 0 -truncated PSD r.v.

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Variant 4. The numbers of units N_k, $k = 1,2, \dots, M$ in the subnets $k = 1,2, \dots, M$ are independent., identically, 0-truncated PSD r.v., and the number M of subnets is 0-truncated PSD r.v., independent of r.v. N_k, $k = 1,2, \dots, M$.

We recall that, according to [1], r.v. Z with values from the set $\{0,1, \dots, n, \dots\}$ is a power series distributed (PSD) r.v. with parameter ϑ and power function $A(\vartheta) = \sum_{k\geq 0} a_z \vartheta^z$ if $P(Z = z) = a_z \vartheta^z / A(\vartheta)$, $a_z \geq 0$, where the power series is convergent with the convergence radius $\tau \in (0,+\infty)$. Shortly, $Z \in$ PSD.

The PSD used in our paper are 0-truncated ones, because the real networks consists from at least one unit. The following assertion assure us that the operation of 0-truncations does not alter the initial quality of distribution to be of PSD class.

Proposition 1 [2]. *If* $Z \in$ PSD *with parameter* ϑ *and power series function* $A(\vartheta) = \sum_{k\geq 0} a_z \vartheta^z$, *then his 0-truncation is a r.v.* $Z^* \in PSD$ *with parameter* ϑ, $\vartheta \in (0, \tau)$, $\tau \in (0,+\infty)$ *and power series function* $A^*(\vartheta) == \sum_{k\geq 1} a_z \vartheta^z = A(\vartheta) - a_0$, *i.e.,* $P(Z = z) = a_z \vartheta^z / A^*(\vartheta)$, $a_z \geq 0$, $z=1,2, \dots$.

Example 1. The following Table 1, from [2], shows the form of PSD parameters of 0-truncated distributions of some classical discrete distributions as $Bin(n; p)$, $Geom(p)$, $Poisson(\lambda)$, $Log(p)$, $NegBin(k; p)$, $Pascal(k; p)$; marked by symbol " *", if their 0-truncation change form as a PSD.

TABLE I. 0-TRUNCATED DISTRIBUTIONS OF SOME CLASSICAL DISCRETE DISTRIBUTIONS

Distribution	a_z	θ	$A(\theta)$	τ
$Bin^*(n;p)$, $n \in \{1,2,..\}$, $0<p<1$	$\begin{cases} \binom{n}{z}, for\ z=\overline{1,n}, \\ 0,\ for\ z=0\ or\ z>n. \end{cases}$	$\frac{p}{1-p}$	$(1+\theta)^n - 1$	$+\infty$
$Poisson^*(\lambda)$, $\lambda > 0$	$\begin{cases} \frac{1}{z!}, for\ z=1,2,... \\ 0,\ for\ z=0. \end{cases}$	λ	$e^{\theta} - 1$	$+\infty$
$Log(p)$, $0<p<1$	$\begin{cases} \frac{1}{z}, for\ z=1,2,... \\ 0,\ for\ z=0. \end{cases}$	p	$-\ln(1-\theta)$	1
$Geom^*(p)$, $0<p<1$	$\begin{cases} 1, for\ z=1,2,... \\ 0,\ for\ z=0. \end{cases}$	$1-p$	$\frac{\theta}{1-\theta}$	1
$NegBin^*(k;p)$, $k \in \{1,2,..\}$, $0<p<1$	$\begin{cases} \binom{z+k-1}{z}, for\ z=1,2,... \\ 0,\ for\ z=0. \end{cases}$	p	$(1-\theta)^{-k} - 1$	1
$Pascal(k;p)$, $k \in \{1,2,..\}$, $0<p<1$	$\begin{cases} \binom{z-1}{k-1}, for\ z=k,k+1,... \\ 0,\ for\ z=\overline{0,k-1}. \end{cases}$	$1-p$	$\left(\frac{\theta}{1-\theta}\right)^k$	1

III. COMPARING THE RELIABILITY OF NETWORK OF TYPE A VS NETWORK OF TYPE B.

We denote by $R_{S\text{-}P}(x)$ the reliability of the type A network and by $R_{P\text{-}S}(x)$ the reliability of the type B network, by the reliability (also called the survival function) of a system understanding the function $R(x) = 1\text{-}L(x)$, where $L(x)$ is the lifetime c.d.f. of this system.

The following two statements highlight the conditions under which the type A network is more reliable than the type B network.

Proposition 2. *For any mathematical model (in variants 1-4), the property of one network to be more reliable than the other network does not depend on lifetime c.d.f.* $F(x)$ *of each unit of the network.*

Proposition 3. *For any mathematical model (in variants 1-4), the series-parallel network is more reliable than the parallel-series network, i.e.,* $R_{S\text{-}P}(x) > R_{P\text{-}S}(x)$ *as soon as* $P(N_1 > M) = \dots = P(N_M > M) = 1$.

In other words, it is sufficient that $P(N_1 > M) = \dots = P(N_M > M) = 1$, which in the case of variant 1 this means that $\min(N_1, N_2, \dots, N_M) > M$, to guarantee that $R_{S\text{-}P}(x) > R_{P\text{-}S}(x)$. Otherwise, this means that not always $R_{S\text{-}P}(x) > R_{P\text{-}S}(x)$. Moreover, Proposition 2 shows that this property takes place regardless of the lifetime c.d.f. $F(x)$ of each unit in the network. So, to illustrate graphically this property is sufficient to take the uniform distribution on [0,1] as the c.d.f. $F(x)$, i.e. $F(x) = xI_{[0,1]}(x) + I_{(1,+\infty)}$, where $I_D(x) = 1$ if $x \in D$, otherwise $I_D(x) = 0$.

For the mathematical models in variants 1-2, our statements being a consequence of the results of the [3], we will bring illustrative examples for variants 3-4.

Example 2. We consider the networks of type A and B according to the mathematical model from variant 3, i.e., the number of units in each subnet is constant and equal to the same number N, while the number of subnets is a r.v. $M \in PSD$ with parameter ω and power series function $B(\omega)$ with radius of convergence $r \in (0, +\infty)$.

As a consequence of Proposition 2, we will consider that lifetime c.d.f. of each unit is given by $F(x) = xI_{[0,1]}(x) + I_{(1,+\infty)}$. Under these conditions it is valid

Proposition 4. *Reliability of type A and B networks in variant 3, when lifetime c.d.f.* $F(x) = xI_{[0,1]}(x) + I_{(1,+\infty)}$ *are given respectively by the functions*

$R_{S\text{-}p}(x) = [B(\omega(1-x^N)) / B(\omega)]\, I_{[0,1]}(x)$ *and*

$R_{P\text{-}S}(x) = [1 - B(\omega(1-(1-x)^N)) / B(\omega)]\, I_{[0,1]}(x)$,

where $B(\omega)$ *is a power series function of r.v.* $M \in PSD$ *with parameter* ω.

So, to exemplify, we will take a few cases when $M \sim Bin^*(n; p)$, i.e. $\omega = p/(1-p)$ and $B(\omega) = (1+\omega)^n - 1$.

N=5, n=3, p=1/2, i.e., $\omega=1$ and $P(N>M)=1$. Then $R_{S\text{-}P}(x) = \{[(1+(1-x^5))^3-1] / (2^3-1)\}\, I_{[0,1]}(x)$ and $R_{P\text{-}S}(x) = \{[(1+(1-(1-x)^5))^3-1] / (2^3-1)\}\, I_{[0,1]}(x)$.

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Their graphical representation below (Fig. 2) shows that

R_{S-P} *(x)* > R_{P-S} *(x)*, which confirms the statement of *Proposition 3*.

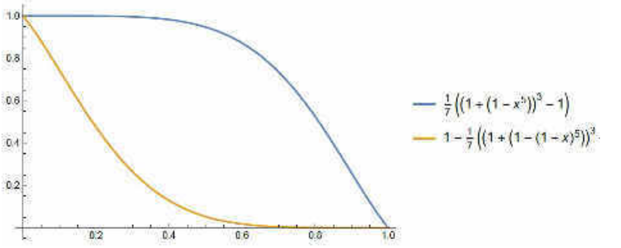

Figure 2.

N=5, n=34, p=1/2, i.e., ω=1 and P(N>M)<1. Then $R_{S-P}(x)=\{[(1+(1-x^5))^{34}-1] / (2^{34}-1)\}\, I_{[0,1]}(x)$ and

$R_{P-S}(x)=\{[(1+(1-(1-x)^5)^{34}-1] / (2^{34}-1)\}\, I_{[0,1]}(x)$.

Their graphical representation below (Fig. 3) shows that inequality R_{S-P} *(x)* > R_{P-S} *(x)* is not sure for all *x*, which also confirms conclusions from the Proposition 3.

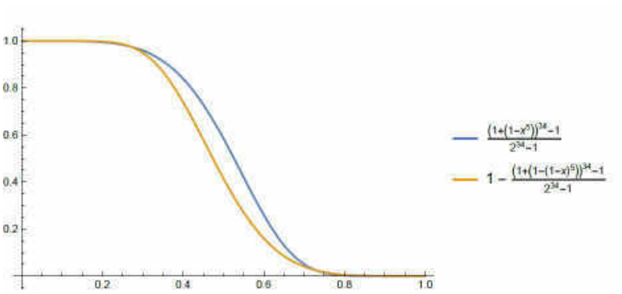

Figure 3.

Moreover, in the following case, R_{S-P} *(x)* < R_{P-S} *(x)*, when

P(N >M) <1.

N=5, n=100, p=1/2, i.e., ω=1. *So,* P(N>M)<1. *Then* $R_{S-P}(x)=\{[(1+(1-x^5))^{100}-1] / (2^{100}-1)\}\, I_{[0,1]}(x)$ and

$R_{P-S}(x)=\{[(1+(1-(1-x)^5)^{100}-1] / (2^{100}-1)\}\, I_{[0,1]}(x)$.

From the graphic representation below (Fig. 4) we see that, indeed, R_{S-P} *(x)* < R_{P-S} *(x)*. This also confirms conclusions from the *Proposition 3*.

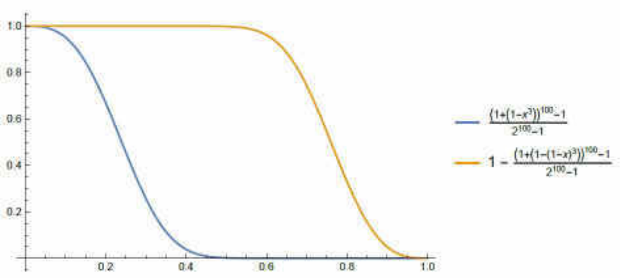

Figure 4.

Example 3. Now, let's consider the networks of type *A* and *B* according to the mathematical models which correspond to variant 4, i.e., the numbers N_i of units in each subnet, *i=1, 2, …* are i.i.d.r.v. of *0*-truncated PSD type with the parameter ϑ and power function *A(ϑ)* with radius of convergence τ ∈ (0, +∞), while the number of subnets *M* is a r.v. independent of numbers N_i , i=1, 2, … and *M* ∈ *PSD* with the parameter *ω* and power series function *B*(ω) with radius of convergence r ∈ (0, + +∞) .

Also, as a consequence of Proposition 2, we will consider that lifetime c.d.f. of each unit is given by $F(x) = =xI_{[0,1]}(x)\,)+I_{(1,+\infty)}$. Under these conditions it is valid

Proposition 5. *Reliability of type A and B networks in variant 4, when lifetime c.d.f.* $F(x)=xI_{[0,1]}(x)\,)+I_{(1,+\infty)}$ *are given respectively by the functions*

$R_{S-P}(x)= [B(\omega(1- A(\vartheta x) / A(\vartheta))) / B(\omega)]\, I_{[0,1]}(x)$ *and*

$R_{P-S}(x)=[1- B(\omega(1- A(\vartheta(1-x)/ A(\vartheta)))) / B(\omega)]\, I_{[0,1]}(x)$,

where A(ϑ) and B(ω) are the power series functions of r.v. N_i *, i=1, 2, …, and M, with parameters ϑ and ω, respectively.*

Now, we consider the following cases.

R.v. N_i ~Pascal*(k;p), i=1, 2, …,M, k=5, p=1/2 and M~Bin*(n; q), n=4, q=1/2. *So, from the Table 1, we deduce that* θ =1/2 *for* p=1/2 *and* ω=1 *for* q=1/2, $A(\theta)=(\theta/1-\theta)^5=1$ for θ =1/2 *and* $B(\omega)=(1+\omega)^4-1=2^4-1$ for ω =1 and $P(N_i>M)=1$, i=1, 2, …, M. *This implies that for* x ∈[0,1]

$R_{S-P}(x)=\{[2-(x/(2-x))^5]^4-1\}/(2^4-1)$ *and*

$R_{P-S}(x)=1-\{[2-((1-x)/(1+x))^5]^4-1\}/(2^4-1)$.

The graphical representation below (Fig. 5) shows that

R_{S-P} *(x)* > R_{P-S} *(x)*, which confirms the statement of *Proposition 3*.

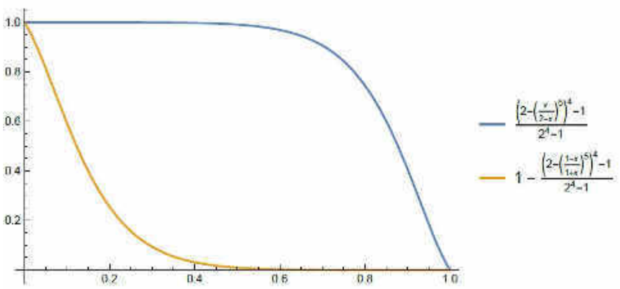

Figure 5.

R.v. N_i ~Bin*(n; q), n=4,q=1/2, i=1, 2, …, and M~Pascal*(k;p), k=5, p=1/2. *So , from the Table 1, we deduce that* θ =1 *for* p=1/2 *and* ω=1/2 *for* q=1/2, $A(\theta)=(1+\theta)^4-1=2^4-1$ for θ =1 *and* $B(\omega)=(\omega/(1-\omega))^5=1$ for ω =1/2 and $P(N_i>M)=0$, i=1, 2, …, M. *This implies that for* x ∈[0,1]

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$R_{S\text{-}P}\,(x)=\{[2^4\text{-}(1+x)^4]\,/\,[2(2^4\text{-}1)\text{-}\,(2^4\text{-}(1+x)^4)]\}^5$ *and*
$R_{P\text{-}S}\,(x)=1\text{-}\{[2^4\text{-}(2\text{-}x)^4]/[2(2^4\text{-}1)\text{-}\,(2^4\text{-}(2^4\text{-}(2\text{-}x)^4)\}^5$.

The graphical representation below (Fig. 6) shows that

$R_{S\text{-}P}\,(x) < R_{P\text{-}S}\,(x)$ because $P(N_i > M)=0,\ i=1,\ 2,\ \ldots,\ M$., which also confirms consequences from the statement of *Proposition 3*.

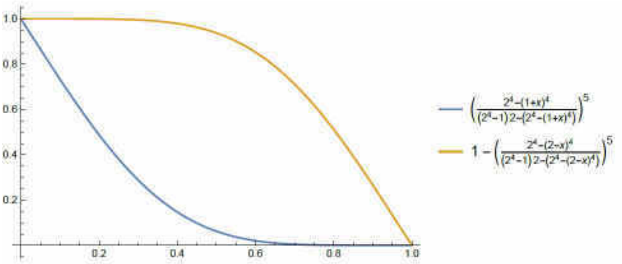

Figure 6.

Finally, also, if there is at least one i for which $P(N_i > M)<1$, do we have that inequality $R_{S\text{-}P}\,(x) > R_{P\text{-}S}\,(x)$ it becomes uncertain. Really

R.v. $N_i \sim Bin^*(n;\,q)$, n=20, q=1/2, i=1, 2, …, and $M \sim Pascal^*(k;p)$, k=100, p=1/2. So , from the Table 1, we deduce that $\theta=1$ for p=1/2 and ω=1/2 for q=1/2, $A(\theta)=(1+\theta)^{20}\text{-}1=2^{20}\text{-}1$ for $\theta=1$ and $B(\omega)=(\omega/(1\text{-}\omega))^{100}=1$ for $\omega=1/2$ and $P(N_i > M)<1$, i=1, 2, …, M. This implies that for $x \in [0,1]$

$R_{S\text{-}P}\,(x)=\{[2^{20}\text{-}(1+x)^{20}]\,/\,[2(2^{20}\text{-}1)\text{-}\,(2^{20}\text{-}(1+x)^{20})]\}^{100}$ and
$R_{P\text{-}S}\,(x)=1\text{-}\{[2^{20}\text{-}(2\text{-}x)^{20}]/[2(2^{20}\text{-}1)\text{-}\,(2^{20}\text{-}(2^{20}\text{-}(2\text{-}x)^{20})\}^{100}$
(Fig. 7).

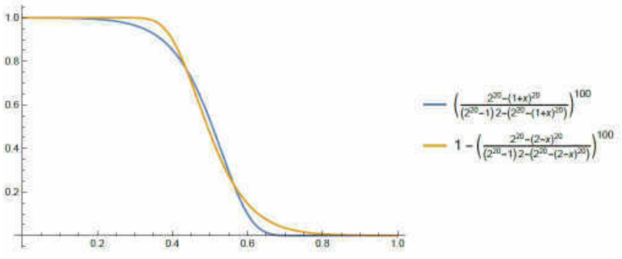

Figure 7.

Conclusions. Comparing the reliability of the serial-parallel type networks with the reliability of the parallel-serial type networks represented in fig. 1 we have shown the following.

Due to the specific / characteristic properties of c.d.f. this comparison does not depend on lifetime c.d.f. of the units that are part of the network.

As a result, we can make the proposed comparison, taking as lifetime c.d.f. F (x) the uniform distribution on the segment [0,1].

We also showed that, finally, in any of the 4 dynamic mathematical models (variants 1-4) the parallel series network is more reliable than the parallel series network as soon as the probability that the number of units in each subnet will be greater than the number of subnets is equal to 1. Otherwise, the comparison is not unambiguous.

This last statement is also confirmed by graphic methods.

Moreover, in the case of mathematical models 3 and 4, the calculation formulas for reliability / survival function are brought.

References

[1] Noack, A., A class of random variables with discrete distribution, Annals of Mathematical Statistics, 1950, Vol.21, No.1, p. 127-132.

[2] Leahu, A., Andrievschi-Bagrin, V., Lifetime distributions and their approximation in reliability of serial/parallel networks, An. Ș t. Univ. Ovidius Constanța, Ser. matematica, Vol. 28(2), 2020, p. 161-172.

[3] Leahu, A., Andrievschi-Bagrin, V., Ciorba,.D., Fiodorov, I., Reliability for the networks with random number of units in each subnet, Abstracts of International Conference Mathematics & IT: Research and Education (MITRE- 2021), Moldova State University, Chişinău, Republic of Moldova, July 01–03, 2021, p.49