https://doi.org/10.52326/ic-ecco.2021/CS.01

The Cauchy problem for quasilinear pseudodifferential equation with integral coefficients

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***Abstract*—We consider a quasilinear pseudodifferential evolution equation with the derivative of order one with respect to the time variable *t* and the pseudodifferential operator this symbol $a_\gamma(\sigma)$, $\sigma \in \mathbb{R}$, homogeneous order $0 < \gamma \le 2$, by space variable *x* with integral coefficients. Such equations describe diffusion on inhomogeneous fractals.**

Keywords*—*Cauchy problem; fractal diffusion; integral coefficient; quasilinear equation

I. Introduction

The first result about solvability of the Cauchy problem for one linear equation of the form (6), (7) were obtained in [3 – 6]. However, completely correct and final, in some cases, results were obtained for one equation, where for the first time the pseudodifferential operator is treated as a hypersingular integral. For systems of such equations, similar results were obtained for the first time in [9, 10]. Professors S.D. Eidelman and Ya.M. Drin are authors of a new direction in mathematics - "Parabolic pseudodifferential equations with non-smooth symbols". These equations are characterized by the fact that their fundamental solutions have power estimates and are proved by them for the first time.

II. Formulation of the Problem

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We should determine the solution of the Cauchy problem

$$u_t(t,x) - A_\gamma u(t,x) + \int_0^1 u(t,\xi)d\xi\, u_x(t,x) = f(t,u),$$

$$(t,x) \in \mathbb{R}_+^2 \equiv \mathbb{R}_+ \times \mathbb{R}, 0 < \gamma \le 2, \quad (1)$$

and initial condition

$$u(0,x) = \varphi(x), x \in \mathbb{R}, \quad (2)$$

where $f : \mathbb{R}_+^2 \to \mathbb{R}$ are known function with such properties [1]

1) *f* is continuous function;

2) for any $t \in \mathbb{R}_+$ and $\{u, u_1, u_2\} \subset \mathbb{R}$ satisfy the conditions

$$| f(t,u)| \le C_1 |u|^{1+\beta}, \quad (3)$$

$$| f(t,u_1) - f(t,u_2)| \le \le c_2 |u_1 - u_2| \max\{|u_1|^\beta, |u_2|^\beta\}, \quad (4)$$

Where c_1, c_2 and β are some positive constants, and $\beta > 1$, A_γ is pseudodifferential operators with symbol $a_\gamma(\sigma)$, $\sigma \in \mathbb{R}$ [2, 7, 8].

We study existence and the only solution of problem (1), (2).

Assume the function $u(t,x)$, $(t,x) \in \mathbb{R}_+^2$, is a solution of the problem (1), (2), and $u(t,x) \in C^{1,1}(\mathbb{R}_+^2)$. A constitute the substitution

$$u(t,x) = v(t,y), \quad (5)$$

$$y = x - \int_0^t \int_0^1 u(\eta,\xi)d\xi d\eta, (t,y) \in \mathbb{R}_+^2.$$

So long as

$$u_t(t,x) = v_t(t,y) + v_y(t,y)\frac{\partial y}{\partial t} =$$

$$= v_t(t,y) - \int_0^1 u(t,\xi)d\xi\, v_y(t,y),$$

$$A_\gamma u(t,x) = A_\gamma v(t,y),$$

then the problem (1), (2) acquire appearance

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$$v_t(t,y) - A_\gamma v(t,y) = f(t, v(t,y)),\quad (t,y) \in \mathbb{R}^2_+, \tag{6}$$

$$v(0,y) = \varphi(y), y \in \mathbb{R}. \tag{7}$$

III. A SOLVING OF THE PROBLEM (6), (7)

The solution of the problem (6), (7) where $v \in C^{1,1}(\mathbb{R}^2_+)$ will be found as

$$v(t,y) = \int_{-\infty}^{\infty} G(t, y-\xi)\varphi(\xi)d\xi + \int_0^t d\tau \int_{-\infty}^{\infty} G(t-\tau, y-\xi) f(\tau, v(t,\xi))d\xi, \quad (t,y) \in \mathbb{R}^2_+, \tag{8}$$

then $G(t,x)$, $(t,x) \in \mathbb{R}^2_+$ is fundamental solution of the pseudodifferential equation [3, 4]

$$\frac{\partial v(t,x)}{\partial t} - A_\gamma v(t,x) = 0, (t,x) \in \mathbb{R}^2_+,$$

and then $\gamma = 1$, $a_\gamma(\sigma) = |\sigma|$, $\sigma \in \mathbb{R}$, and $G(t,x) = \frac{1}{\pi}\frac{t}{t^2 + x^2}$, $(t,x) \in \mathbb{R}^2_+$.

Integral equation (8) we unbind by consecutive approximate method:

$$v_0(t,y) = \int_{-\infty}^{+\infty} G(t, y-\xi)\varphi(\xi)d\xi,$$

$$v_k(t,y) = v_0(t,y) + \int_0^t d\tau \int_{-\infty}^{\infty} G(t-\tau, y-\xi) \times f(\tau, v_{k-1}(\tau,\xi))d\xi, k \ge 1, (t,y) \in \mathbb{R}^2_+. \tag{9}$$

We denote $H(\mathbb{R}^2_+)$ a space of continuously functions where $\langle \cdot \rangle_{\mathbb{R}^2_+}$ denotes norm in this space [11]

$$\langle \psi(t,x) \rangle_{\mathbb{R}^2_+} \equiv \sup_{(t,x)\in\mathbb{R}^2_+} \frac{|\psi(t,x)|}{G(t+\chi, x)},$$

where χ is a positive number.

Obviously what

$$|f(t, v(t,y))| \le \le C_1 \frac{|v(t,y)|^{1+\beta}}{G^{1+\beta}(t+\chi, y)} G^{1+\beta}(t+\chi, y) \le \le C_1 \langle v(t,y) \rangle^{1+\beta}_{\mathbb{R}^2_+} G^{1+\beta}(t+\chi, y), (t,y) \in \mathbb{R}^2_+. \tag{10}$$

At that time

$$|(Mv)(t,y)| \equiv \equiv \left| \int_0^t d\tau \int_{-\infty}^{\infty} G(t-\tau, y-\xi) f(\tau, v(\tau,\xi))d\xi \right| \le$$

$$\le c_1 \int_0^t d\tau \int_{-\infty}^{\infty} \langle v \rangle^{1+\beta}_{\mathbb{R}^2_+} G^{1+\beta}(\tau+\chi, \xi) \times G(t-\tau, y-\xi)d\xi \le$$

$$\le c_1 \langle v \rangle^{1+\beta}_{\mathbb{R}^2_+} \int_0^{\infty} \frac{d\tau}{\pi^\beta (\tau+\chi)^\beta} \times \int_{-\infty}^{\infty} G(t-\tau, y-\xi) G(\tau+\chi, \xi)d\xi =$$

$$= c_3 \langle v \rangle^{1+\beta}_{\mathbb{R}^2_+} G(t+, y) \left. \frac{(\tau+\chi)^{1+\beta}}{1-\beta} \right|_0^{\infty} = = c_4 \langle v \rangle^{1+\beta}_{\mathbb{R}^2_+} G(t+\chi, y), \tag{11}$$

where $1-\beta < 0$ or $\beta > 1$, $c_4 = c_3 \chi^{1-\beta}(\beta-1)^{-1}$, and use a roll up formula

$$\int_{-\infty}^{\infty} G(t-\tau, y-\xi) G(\tau+\chi, \xi)d\xi = = G(t+\chi, y), \tau < t, y \in \mathbb{R}.$$

So from inequality (11) we obtain inequality

$$\langle Mv \rangle_{\mathbb{R}^2_+} \le C_4 \langle v \rangle^{1+\beta}_{\mathbb{R}^2_+}. \tag{12}$$

Further, for the arbitrary $\{v, w\} \subset H(\mathbb{R}^2_+)$ like that $\langle v \rangle_{\mathbb{R}^2_+} \le K$, $\langle w \rangle_{\mathbb{R}^2_+} \le K$ we have

$$|(Mv - Mw)(t,y)| = = \left| \int_0^t d\tau \int_{-\infty}^{\infty} G(t-\tau, y-\xi)[f(\tau, v(\tau,\xi)) - - f(\tau, w(\tau,\xi))]d\xi \right| \le$$

$$\le C_2 \int_0^t d\tau \int_{-\infty}^{\infty} G(t-\tau, y-\xi) \times \max\{|v|^\beta, |w|^{eta}\} |v - w| \times$$

$$\times G^{1+\beta}(t+\chi, \xi) G^{-(1+\beta)}(\tau+\chi, \xi)d\xi \le$$

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$$\le C_2\int_0^t d\tau\int_{-\infty}^{\infty}\max\{(\frac{|v(\tau,\xi)|}{G(\tau+\chi,\xi)})^{\beta},(\frac{|w(\tau,\xi)|}{G(\tau+\chi,\xi)})^{\beta}\}\times$$

$$\times\frac{|v(\tau,\xi)-w(\tau,\xi)|}{G(\tau+\chi,\xi)}G(t-\tau,y-\xi)\times$$

$$\times G^{1+\beta}(\tau+\chi,\xi)d\xi\le$$

$$\le C_2K^{\beta}\langle v-w\rangle_{\mathbb{R}_+^2}\frac{1}{\pi^{\beta}}\int_0^{\infty}\frac{d\tau}{(\tau+\chi)^{\beta}}\times$$

$$\times\int_{-\infty}^{\infty}G(t-\tau,y-\xi)G(\tau+\chi,\xi)d\xi=$$

$$=C_5K^{\beta}\langle v-w\rangle_{\mathbb{R}_+^2}G(t+\chi,y),(t,y)\in\mathbb{R}_+^2,$$

and we obtain inequality

$$\langle Mv-Mw\rangle_{\mathbb{R}_+^2}\le C_5K^{\beta}\langle v-w\rangle_{\mathbb{R}_+^2}. \quad (13)$$

We estimate the norm $\langle v_0\rangle_{\mathbb{R}_+^2}$. We have

$$|v_0(t,y)|=$$

$$=|\int_{-\infty}^{\infty}G(t,y-\xi)G(\chi,\xi)\frac{\varphi(\xi)}{G(\chi,\xi)}d\xi|\le$$

$$\le\langle\varphi\rangle_{\mathbb{R}}G(t+\chi,y),(t,y)\in\mathbb{R}_+^2$$

and that is why

$$\langle v_0\rangle_{\mathbb{R}_+^2}\le\langle\varphi\rangle_{\mathbb{R}}. \quad (14)$$

Assume then the function φ is a continual and $\langle\varphi\rangle_{\mathbb{R}}<\delta$ and have a small $\delta>0$.

At that time from (9) and (12) swim out, what

$$\langle v_R\rangle_{\mathbb{R}_+^2}\le\delta+c_4\langle v_{k-1}\rangle_{\mathbb{R}_+^2}^{1+\beta},k\ge1. \quad (15)$$

Therefore

$$\langle v_1\rangle_{\mathbb{R}_+^2}\le c_4\delta^{1+\beta}+\delta\equiv N_1(\delta),$$

$$\langle v_2\rangle_{\mathbb{R}_+^2}\le\delta+c_4N_1^{1+\beta}(\delta)\equiv N_2(\delta),$$

$$\ldots\ldots\ldots\ldots\ldots$$

$$\langle v_k\rangle_{\mathbb{R}_+^2}\le\delta+c_4N_{k-1}^{1+\beta}(\delta)\equiv N_k(\delta),k\ge1. \quad (16)$$

From (16) swim out, that to small δ

$$\langle v_k\rangle_{\mathbb{R}_+^2}\le N(\delta),k\ge1, \quad (17)$$

and $N(\delta)\to0$ if $\delta\to0$.

From the inequalities (13) and (17) we obtain

$$\langle v_k-v_{k-1}\rangle_{\mathbb{R}_+^2}=\langle Mv_{k-1}-Mv_{k-2}\rangle_{\mathbb{R}_+^2}\le$$

$$\le c_2N^{\beta}(\delta)\langle v_{k-1}-v_{k-2}\rangle_{\mathbb{R}_+^2},k\ge2.$$

Let's assume that exists a constant $\delta>0$ that $c_5N^{\beta}(\delta)<1$. The sequences $\{v_k,k\ge1\}$ is a component sum to the functional row

$$v_0+(v_1-v_0)+\ldots+(v_k-v_{k-1})+\ldots$$

what valuing (in norm space $H(\mathbb{R}_+^2)$) to the number row. Then exist $\lim_{k\to\infty}v_k=v\in H(\mathbb{R}_+^2)$, and v is continuous function in $\mathbb{R}_+^2$. If in (9) $k\to\infty$ therefore v is solution of the equation (8).

IV. The Main Result

Theorem Assume that the function f is a continuity and satisfy the conditions (3) and (4). Then exists the numbers $\delta>0$, $c>0$ so if $\varphi\in H(\mathbb{R})$ and $\langle\varphi\rangle_{\mathbb{R}^2}\le\delta$, then exist only solution u of the problem (1), (2), where $u\in H(\mathbb{R}_+^2)$ and $\langle u\rangle_{\mathbb{R}_+^2}\le c$.

V. Conclusion

Fractional calculus is a branch of mathematics that with the study of integrals and derivatives of non-integer orders, plays an outstanding role and have found several applications in large areas research during the last decade. Behaviour of many dynamical systems can be described and studied using the fractional order system. Fractional derivatives cribe effect of memory.

The Cauchy problem (1), (2) described diffusion on inhomogeneous fractals.

The equation (1) is essentially nonliear and solution of the problem (1), (2) can be investigated by numerical methods.

The investigated problem can be generalized in case $n>1$ variables, as well as variables depending on t or t, x, $t>0, x\in\mathbb{R}^n$.

For this it is necessary to spend appropriate research.

The fast result about solvability of the Caucy problem for linear equation in form (6) were obtained by S.D. Eidelman and Y.M. Drin, who are authors of a new direction of mathematics - "Parabolic pseudodifferential equations with non-smooth symbols".

The problem of physical content and admissibility of differentiation of fractional order is considered in [12]. The generalizing character of the apparatus of derivative and integrals of fractional order, it is demonstrated in describing the model of a medium with the elasticity viscoelasticity-viscous liquid properties.

https://doi.org/10.52326/ic-ecco.2021/CS.01

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