Computer Science Journal of Moldova, vol.**23**, no.1(67), 2015

Maximum Inaccuracies of Second Order *

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Abstract

Let an indirectly measurable variable Y be represented as a function of finite number of directly measurable variables X_1, $X_2, ..., X_n$. We introduce maximum absolute and relative inaccuracies of second order of Y – this idea is a continuation of our research of a new principle for representing the maximum inaccuracies of Y using the inaccuracies of $X_1, X_2, ..., X_n$. Using inaccuracies of second order we determine the maximum inaccuracies of indirectly measurable variable Y with quadratic approximation which gives their values more precisely. We give algorithmically an easily applicable method for determining their numerical values. The defined by us maximum inaccuracies of second order give the opportunity for more precise determination of the inaccuracy when measuring indirectly measurable variables.

Keywords: indirectly measurable variable; maximum inaccuracy; Taylor series expansion.

1 Introduction

When conducting an experiment each measurement has imperfections which cause an error in the obtained result for the given measurement. We believe that it is more accurate not to speak of an error but of an inaccuracy of the measurement – we consider *the accuracy* of the measurement as a quality which can be represented quantitatively.

* The current research is done with the financial support of the Fund "Scientific Studies" of the Bulgarian Ministry of Education, Youth and Science as part of the contract TO1/2

Following [1], we assume that a *principle of measurement* is the theoretical basis of the measurement; a *method of measurement* is the logical chain of operations in order to conduct the measurement; and a *procedure of measurement* is the set of operations described in details and specifics which are used in order to conduct the concrete measurements given by the corresponding method of measurement.

Let X be a directly measurable variable which in k observations of an experiment assumes values $x_1, x_2, ..., x_k$ whose arithmetic mean is $\bar{x} = \frac{1}{k}\sum_{i=1}^{k} x_i$. The measurement of the values of X is accompanied by inaccuracies which by their *way of expression* are absolute or relative [2], [3], [4], [5], [6].

The absolute inaccuracy ΔX is represented in the unit of the measured variable. The value of the absolute inaccuracy at the i-th observation is $\Delta x_i = x_i - \bar{x}$.

The relative inaccuracy $\frac{\Delta X}{X}$ is a dimensionless variable. The value of the relative inaccuracy at the i-th observation is $\left|\frac{\Delta x_i}{x_i}\right| = \left|\frac{x_i - \bar{x}}{x_i}\right|$ ($x_i \neq 0$).

By their *character of change* the inaccuracies are random or systematic [2], [3], [4], [5], [6]. The nature and physical meaning of the random and systematic inaccuracies of a measurement are different.

The random (or *stochastic*) *inaccuracy* is a result of either unknown, or unforeseeable, or known variable time and/or space effects of influencing variables. It is assumed that under fixed conditions of the experiment (conducted using the same methods and the same utensils) this inaccuracy is volatile and is not regularly altered. It can have different values with each measurement. Although the random inaccuracy of the result of a measurement cannot be compensated, usually it can be reduced by increasing the number of observations.

The systematic inaccuracy is a result of known constant influences on the environment of the experiment. It is assumed that under fixed conditions of the experiment (conducted using the same methods and with the same utensils) this inaccuracy is constant or obeys certain regularities. It either has equal values under all measurements, or these values can be determined experimentally, or they can be calculated. The systematic inaccuracy similarly to the random one cannot be eliminated but can often be reduced.

The inaccuracy of the result of a measurement shows the lack of knowledge regarding the exact value of the measured variable. Thus, the result of a measurement, even after increasing the number of observations is still only an *approximation* of the value of the measured variable.

Practically, there are many different sources of inaccuracy in a measurement, including:

- incomplete definition or incomplete modelling of the measured variable;
- approximations or assumptions used in the method or in the measurement procedure;
- inadequate knowledge regarding the influence of the environment on the measurement;
- deviation when reading the utensils or the presence of distinction threshold for the values of the utensils;
- inaccurate values of constants or other parameters given by measurement standards, reference materials or other external sources.

These sources are not necessarily independent and some of them could introduce inaccuracies in others.

Let us also point out that the categorisation of the inaccuracies as 'random' and 'systemic' can also be ambiguous. For example, a 'random' inaccuracy in one measurement can become a 'systematic' inaccuracy in another measurement, in which the result of the first one is used as input data.

Let Y be an indirectly measurable variable which depends explicitly on n directly measurable variables which are modelled (with the help of measurement utensils or methods) by n real independent variables

$X_1, X_2, ..., X_n$. Let us denote with f the real function with arguments $X_1, X_2, ..., X_n$ which helps represent Y, i.e. $Y = f(X_1, X_2, ..., X_n)$. Moreover, let f be defined in a given neighbourhood of the point $(X_1, X_2, ..., X_n)$ and in this neighbourhood there are continuous partial derivatives up to order m ($m \in \mathbb{N}$) with respect to all its variables. Then [7], [8] we can expand f in Taylor series around the point $(X_1, X_2, ..., X_n)$:

$$\begin{aligned} f(X_1 + dX_1, X_2 + dX_2, ..., X_n + dX_n) - f(X_1, X_2, ..., X_n) = \\ = \sum_{i=1}^{n} \frac{\partial f}{\partial X_i} dX_i + \frac{1}{2!} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial X_i \partial X_j} dX_i dX_j + ... + R_m. \end{aligned} \tag{1}$$

In (1) all the partial derivations are calculated in the point $(X_1, X_2, ..., X_n)$ and the remainder R_m is calculated in the point $(X_1 + \theta dX_1, X_2 + \theta dX_2, ..., X_n + \theta dX_n)$, where $0 < \theta < 1$, and in its different forms it satisfies certain conditions [7], [8].

In order to determine the maximum inaccuracies of an indirectly measurable variable the primarily used methods are statistical. In [9], [10], [11] we developed a new method for studying in linear approximation the maximum inaccuracies of an indirectly measurable variable $Y = f(X_1, X_2, ..., X_n)$. Using it we represent the maximum inaccuracies as hyperplanes in metric spaces and define dimensionless scale describing the quality of an experiment.

The purpose of the current paper is to continue these studies by introducing the maximum inaccuracies of the second order of f based on the expansion (1). Using them we determine the maximum inaccuracies of the indirectly measurable variable Y of quadratic approximation. The quadratic approximation gives the values of inaccuracies more precisely. Moreover, we give in algorithmic form an easily applicable method for determining their numerical values and give a practical example showing its application.

Our new method for finding the maximum inaccuracies of indirectly measureable variable adds to and develops the classical method. The main advantages of our method are the high precision and applicability to wider class of functions.

2 Principle for representing in linear approximation the maximum inaccuracies of an indirectly measurable variable

Let $Y = f(X_1, X_2, ..., X_n)$ be an indirectly measurable variable depending on the directly measurable variables $X_1, X_2, ..., X_n$. Moreover, let us have k_i observations of the directly measurable variable X_i in a given experiment, when the values, respectively, $x_{i1}, x_{i2}, ..., x_{ik_i}$ $(i = 1, 2, ..., n)$ are measured. The value of the partial derivative $\dfrac{\partial f}{\partial X_i}$ calculated at the m-th observation we denote with $\dfrac{\partial f}{\partial x_{im}}$ $(m = 1, 2, ..., k_i)$ and the arithmetic mean of the absolute values of the partial derivative $\dfrac{\partial f}{\partial X_i}$ $(i = 1, 2, ..., n)$ we denote with $\overline{\left|\dfrac{\partial f}{\partial x_i}\right|} = \dfrac{1}{k_i}\sum_{m=1}^{k_i}\left|\dfrac{\partial f}{\partial x_{im}}\right|$. Similarly, we denote with

$$\overline{\left|\frac{x_i}{f}\cdot\frac{\partial f}{\partial x_i}\right|} = \frac{1}{k_i}\sum_{m=1}^{k_i}\left|\frac{x_{im}}{f}\cdot\frac{\partial f}{\partial x_{im}}\right|$$

the arithmetic mean of the absolute values of $\dfrac{X_i}{f}\cdot\dfrac{\partial f}{\partial X_i}$ $(i = 1, 2, ..., n)$.

Then according to [9] the maximum absolute inaccuracy (MAI) of the indirectly measurable variable Y can be given in linear approximation as a function

$$\Delta^1 Y = \Delta^1 f = \sum_{i=1}^{n}\overline{\left|\frac{\partial f}{\partial x_i}\right|}\cdot|\Delta X_i| \tag{2}$$

of the absolute values of the absolute inaccuracies $\Delta X_1, \Delta X_2, ..., \Delta X_n$ of the directly measurable variables $X_1, X_2, ..., X_n$. Moreover, the coefficients $\overline{\left|\dfrac{\partial f}{\partial x_1}\right|}, \overline{\left|\dfrac{\partial f}{\partial x_2}\right|}, ..., \overline{\left|\dfrac{\partial f}{\partial x_n}\right|}$ we assume to be *constants* (within the experiment), and the absolute inaccuracies $\Delta X_1, \Delta X_2, ..., \Delta X_n$ themselves we assume to be *variables*.

According to [11] the maximum relative inaccuracy (MRI) of the indirectly measurable variable Y can be given in linear approximation as a function

$$\frac{\Delta^1 Y}{Y} = \frac{\Delta^1 f}{|f|} = \sum_{i=1}^{n} \overline{\left|\frac{x_i}{f} \cdot \frac{\partial f}{\partial x_i}\right|} \cdot \left|\frac{\Delta X_i}{X_i}\right| \tag{3}$$

of the relative inaccuracies $\frac{\Delta X_1}{X_1}, \frac{\Delta X_2}{X_2}, ..., \frac{\Delta X_n}{X_n}$ of the directly measurable variables $X_1, X_2, ..., X_n$. Moreover, the coefficients $\overline{\left|\frac{x_1}{f} \cdot \frac{\partial f}{\partial x_1}\right|}$, $\overline{\left|\frac{x_2}{f} \cdot \frac{\partial f}{\partial x_2}\right|}, ..., \overline{\left|\frac{x_n}{f} \cdot \frac{\partial f}{\partial x_n}\right|}$ are assumed to be *constants* (within the experiment) and the relative inaccuracies $\frac{\Delta X_1}{X_1}, \frac{\Delta X_2}{X_2}, ..., \frac{\Delta X_n}{X_n}$ themselves are assumed to be *variables*.

In the classical theory of errors the coefficients $\left|\frac{\partial f}{\partial X_1}\right|, \left|\frac{\partial f}{\partial X_2}\right|, ...,$ $\left|\frac{\partial f}{\partial X_n}\right|$ are called *coefficients of influence* of the absolute inaccuracies $\Delta X_1, \Delta X_2, ..., \Delta X_n$ in MAI [5]. Similarly, we call the coefficients $\left|\frac{X_1}{f} \cdot \frac{\partial f}{\partial X_1}\right|, \left|\frac{X_2}{f} \cdot \frac{\partial f}{\partial X_2}\right|, ..., \left|\frac{X_n}{f} \cdot \frac{\partial f}{\partial X_n}\right|$ *coefficients of influence* of the relative inaccuracies $\frac{\Delta X_1}{X_1}, \frac{\Delta X_2}{X_2}, ..., \frac{\Delta X_n}{X_n}$ in MRI.

Theoretically, a real function can converge to infinite values. Whatever the experiment, however, the measurement utensils measure physically real and not mathematically modelled variables thus showing that the measured values are always finite. The ideas of a jump in the behaviour of a real variable or of its possible infinite value are only theoretical (model) – they are sensible abstractions when describing the objective reality. This is why however fast the change of a function describing the behaviour of a real variable is, it is never jumpy but rather smooth for every small enough time interval. Therefore, given a real experiment the values of the coefficients of influence change in a small enough interval and thus can be estimated accurately enough using their arithmetic mean. This is why the representations (2) and (3) are reasonable and correct.

3 Representation in quadratic approximation the maximum inaccuracies of an indirectly measurable variable

Let $Y = f(X_1, X_2, ..., X_n)$ be an indirectly measurable variable which is defined as in (1). When nonlinearity of f is considerable in order to determine the numerical values of the maximum inaccuracies of Y more adequately we consider the expansion of the function in Taylor series to terms of the second order.

Then we can determine MAI of Y using

$$\Delta Y = \sum_{i=1}^{n} \left| \frac{\partial f}{\partial X_i} \right| \cdot |\Delta X_i| + \frac{1}{2} \sum_{i,j=1}^{n} \left| \frac{\partial^2 f}{\partial X_i \partial X_j} \right| \cdot |\Delta X_i| \cdot |\Delta X_j| \,,$$

and MRI of Y using

$$\frac{\Delta Y}{Y} = \frac{1}{Y} \left(\sum_{i=1}^{n} \left| \frac{\partial f}{\partial X_i} \right| \cdot |\Delta X_i| + \frac{1}{2} \sum_{i,j=1}^{n} \left| \frac{\partial^2 f}{\partial X_i \partial X_j} \right| \cdot |\Delta X_i| \cdot |\Delta X_j| \right).$$

Let

$$\Delta^2 Y = \Delta^2 f = \sum_{i,j=1}^{n} \overline{\left| \frac{\partial^2 f}{\partial x_i \partial x_j} \right|} \cdot |\Delta X_i| \cdot |\Delta X_j| \qquad (4)$$

and

$$\frac{\Delta^2 Y}{Y} = \frac{\Delta^2 f}{|f|} = \sum_{i,j=1}^{n} \overline{\left| \frac{x_i x_j}{f} \cdot \frac{\partial^2 f}{\partial x_i \partial x_j} \right|} \cdot \left| \frac{\Delta X_i}{X_i} \right| \cdot \left| \frac{\Delta X_j}{X_j} \right|. \qquad (5)$$

MAI and MRI, defined respectively in (2) and (3), we call *maximum absolute inaccuracy of first order* and *maximum relative inaccuracy of first order*. The variables, defined respectively in (4) and (5), we call *maximum absolute inaccuracy of second order* and *maximum relative inaccuracy of second order.*

So with quadratic approximation MAI ΔY of the indirectly measurable variable Y can be presented in the form

$$\Delta Y = \Delta^1 Y + \frac{1}{2}\Delta^2 Y, \tag{6}$$

and MRI $\frac{\Delta Y}{Y}$ of Y – in the form

$$\frac{\Delta Y}{Y} = \frac{\Delta^1 Y}{Y} + \frac{1}{2}\frac{\Delta^2 Y}{Y}. \tag{7}$$

Let us point out that when needed one could introduce similarly MAI and MRI of n-th order for an arbitrary natural number n ($n > 2$). Moreover, with n-th approximation MAI ΔY of the indirectly measurable variable Y will be of the form $\Delta Y = \Delta^1 Y + \frac{1}{2}\Delta^2 Y + ... + \frac{1}{n!}\Delta^n Y$ and MRI $\frac{\Delta Y}{Y}$ of Y will be of the form $\frac{\Delta Y}{Y} = \frac{\Delta^1 Y}{Y} + \frac{1}{2}\frac{\Delta^2 Y}{Y} + ... + \frac{1}{n!}\frac{\Delta^n Y}{Y}$.

4 Method for determining the numerical values of the maximum inaccuracies of second order of an indirectly measurable variable

Let us have k_i observations made in an experiment for the directly measurable variables $X_1, X_2, ..., X_n$ as $x_{i1}, x_{i2}, ..., x_{ik_i}$ are the values measured for the variable X_i ($i = 1, 2, ..., n$).

In [9], [10], [11] we give a method for determining the numerical values of the maximum inaccuracies of first order of an indirectly measurable variable. In this section we further develop this method similarly for inaccuracies of second order.

The determining of the numerical value of MAI of second order of an indirectly measurable variable as logical sequence of operations can be described algorithmically in the following steps:

- For $m = 1, 2, ..., k_i,\ l = 1, 2, ..., k_j$ the values of the coefficients $\left|\dfrac{\partial^2 f}{\partial X_{im} \partial X_{jl}}\right|$ for all $i, j = 1, 2, ..., n$ and the absolute inaccuracies $\Delta X_{1m}, \Delta X_{2m}, .., \Delta X_{nm}$ are calculated.

- The arithmetic means $\overline{\left|\dfrac{\partial^2 f}{\partial x_i \partial x_j}\right|} = \dfrac{1}{k_i k_j} \sum_{m=1}^{k_i} \sum_{l=1}^{k_j} \left|\dfrac{\partial^2 f}{\partial x_{im} \partial x_{jl}}\right|$ of the absolute values of the partial derivatives for all $i, j = 1, 2, ..., n$ are found.

- By (4), given constant values of $\overline{\left|\dfrac{\partial^2 f}{\partial x_i \partial x_j}\right|}$, we get the representation of MAI $\Delta^2 Y$ of second order of the indirectly measurable variable Y.

- The numerical value $\Delta^2 y$ of MAI $\Delta^2 Y$ is determined by applying $\Delta^2 y = \Delta^2 Y \left(\overline{\Delta x_1}, \overline{\Delta x_2}, ..., \overline{\Delta x_n}\right)$.

Similarly, the method can be applied for determining the numerical value $\dfrac{\Delta^2 y}{y}$ of MRI $\dfrac{\Delta^2 Y}{Y}$ of the indirectly measurable variable Y.

Having determined $\Delta^1 Y$, $\Delta^2 Y$ and $\dfrac{\Delta^1 Y}{Y}$, $\dfrac{\Delta^2 Y}{Y}$, the numerical values of the quadratic approximations of the maximum inaccuracies ΔY and $\dfrac{\Delta Y}{Y}$ can be determined according to formulas (6) and (7).

The most substantial difference between the maximum inaccuracies of first and of second order is that with respect to its arguments the inaccuracies of first order are additive, whereas the inaccuracies of second order are also multiplicative. This multiplicativity of the introduced by us instrument for calculation shows that the inaccuracies of second order are more sensitive than the inaccuracies of first order. The main contribution in the numerical value of the maximum inaccuracies is given by the inaccuracies of first order. However, when these values are practically identical, the inaccuracies of second order can give additional and more precise information regarding which experiment is more accurate.

In [11] it is described how the method for determining the numerical values of the maximum inaccuracies of first order of an indirectly measurable variable can also be applied to continuous functions which do not have first derivative with respect to some of its arguments in given points. Similarly, this method can also be applied in order to determine the numerical values of the maximum inaccuracies of second order and to functions which have continuous first derivatives but do not have a second derivative with regards to some of its arguments in given points. Indeed, let for some i, $1 \leq i \leq n$, the derivative $\dfrac{\partial f}{\partial X_i}(X_1, X_2, ..., X_n)$ not be differentiable with respect to its argument X_k, $1 \leq k \leq n$, in given points a_{kj} but is continuous with respect to this argument in a_{kj}, where the index j assumes a finite number of values. Then the derivative $\dfrac{\partial^2 f}{\partial X_i \partial X_k}$ does not exist in the points a_{kj}. For $\dfrac{\partial f}{\partial X_i}$, however, there are both right and left derivatives when $X_k \to a_{kj}$. Then the representation of MAI (respectively, MRI) of second order in the formula (4) (respectively, formula (5)) can be determined by the limit for which the coefficient $\left|\dfrac{\partial^2 f}{\partial X_i \partial X_k}\right|$ (respectively, $\left|\dfrac{X_i X_k}{f}\dfrac{\partial^2 f}{\partial X_i \partial X_k}\right|$) has greater value of the two. (Let us point out that the values of the two limits have to be different, otherwise $\dfrac{\partial f}{\partial X_i}$ would be differentiable with respect to X_k.)

5 Example

It is known [3] that the temperature coefficient α_t of the electrical resistance of the metal of the resistance thermometer given temperature t can be determined from the formula $R_t = R_{t_0}[1 + \alpha_t(t - t_0)]$. The variables R_t and R_{t_0} are respectively the resistance of the resistance thermometer given temperature t and given initial temperature t_0. Thus the coefficient α_t can be represented as an indirectly measurable variable depending on the directly measurable variables t, t_0, R_t,

R_{t_0} as follows:

$$\alpha_t = \frac{1}{t - t_0}\left(\frac{R_t}{R_{t_0}} - 1\right).$$

In two different experiments, each of which has ten observations, the resistance of a platinum resistance thermometer Pt100, Type 404, given certain temperature has been measured and the data has been summarised in Table 1. The values of the arithmetic mean of the temperature coefficient α_t for the both experiments, based on the data from Table 1, are respectively $\alpha_t(1) = 0.003741C^{-1}$ and $\alpha_t(2) = 0.003742C^{-1}$. In order to find out which of the two values is calculated with better accuracy we will compare their respective MRI.

Table 1. Values of the temperature coefficient α_t of resistance thermometer Pt100, Type 404, given certain values of the temperature t and the resistance R_t

First experiment				Second experiment			
Observ. No.	t $[C]$	R_t $[\Omega]$	α_t $[C^{-1}]$	Observ. No.	t $[C]$	R_t $[\Omega]$	α_t $[C^{-1}]$
1	24.9	109.69	0.00374	1	24.9	109.7	0.003746
2	24.9	109.69	0.00374	2	24.9	109.7	0.003746
3	24.9	109.69	0.00374	3	24.9	109.7	0.003746
4	25	109.73	0.003741	4	24.9	109.7	0.003746
5	25	109.73	0.003741	5	25	109.73	0.003741
6	25	109.73	0.003741	6	25	109.73	0.003741
7	25	109.73	0.003741	7	25	109.73	0.003741
8	25.1	109.77	0.003741	8	25	109.73	0.003741
9	25.1	109.77	0.003741	9	25.1	109.76	0.003735
10	25.1	109.77	0.003741	10	25.1	109.76	0.003735

According to (2) in linear approximation MAI of α_t is

$$\Delta^1\alpha_t = \overline{\left|\frac{\partial\alpha_t}{\partial t}\right|}\cdot|\Delta t| + \overline{\left|\frac{\partial\alpha_t}{\partial t_0}\right|}\cdot|\Delta t_0| + \overline{\left|\frac{\partial\alpha_t}{\partial R_t}\right|}\cdot|\Delta R_t| + \overline{\left|\frac{\partial\alpha_t}{\partial R_{t_0}}\right|}\cdot|\Delta R_{t_0}|.$$

The absolute inaccuracies Δt_0 and ΔR_{t_0} are in fact the systematic inaccuracies with which the thermometer and the ohmmeter used in the experiments are calibrated to work. The actual contribution of these inaccuracies to MAI $\Delta^1\alpha_t$ is negligibly small, thus we can assume that $\Delta t_0 = 0$ and $\Delta R_{t_0} = 0$. Then

$$\Delta^1\alpha_t = \overline{\left|\frac{\partial\alpha_t}{\partial t}\right|} \cdot |\Delta t| + \overline{\left|\frac{\partial\alpha_t}{\partial R_t}\right|} \cdot |\Delta R_t|\,,$$

i.e.

$$\Delta^1\alpha_t = \overline{\left|\frac{1}{(t-t_0)^2}\left(\frac{R_t}{R_{t_0}} - 1\right)\right|} \cdot |\Delta t| + \overline{\left|\frac{1}{(t-t_0)} \cdot \frac{1}{R_{t_0}}\right|} \cdot |\Delta R_t|\,.$$

Let us determine the accuracy of the first experiment by determining the numerical value of MAI $\Delta^1\alpha_t$. It is known that given an initial temperature $t_0 = 10^oC$ the value of the resistance of the resistance thermometer is $R_{t_0} = 103.9\Omega$. Then the representation of MAI (precise to three decimal digits) is $\Delta^1\alpha_t = 2.494 \times 10^{-4}\,|\Delta t| + 6.416 \times 10^{-4}\,|\Delta R_t|$. Then, according to Section 4, the value of MAI $\Delta^1\alpha_t$ in the first experiment is $\Delta^1\alpha_t(1) = 3.036 \times 10^{-6}C^{-1}$. This allows us to determine the value of MRI of the first experiment $\dfrac{\Delta^1\alpha_t}{\alpha_t}(1) = 0.008$. (Let us point out that the value of MRI $\dfrac{\Delta^1\alpha_t}{\alpha_t}$ can also be found directly from formula (3).)

Similarly, for the second experiment the representation of MAI is $\Delta^1\alpha_t = 2.558 \times 10^{-4}\,|\Delta t| + 6.425 \times 10^{-4}\,|\Delta R_t|$, its value is $\Delta^1\alpha_t(2) = = 2.871 \times 10^{-6}C^{-1}$ and the value of MRI is $\dfrac{\Delta^1\alpha_t}{\alpha_t}(2) = 0.008$.

Due to $\dfrac{\Delta^1\alpha_t}{\alpha_t}(1) = \dfrac{\Delta^1\alpha_t}{\alpha_t}(2) = 0.008$ we cannot make a conclusion for the accuracy of the measurement of the temperature coefficient α_t by using MRI of first order. Thus we will use MRI of second order.

According to (3) MAI $\Delta^2\alpha_t$ of second order has the following form:

$$\Delta^2\alpha_t = \overline{\left|\frac{2}{(t-t_0)^3}\left(\frac{R_t}{R_{t_0}} - 1\right)\right|} \cdot |\Delta t|^2 + \overline{\left|\frac{1}{(t-t_0)^2 R_{t_0}}\right|} \cdot |\Delta t| \cdot |\Delta R_t|\,.$$

The corresponding representations of MAI of second order for the first and the second experiment are

$$\Delta^2\alpha_t(1) = 3.345 \times 10^{-5} \, |\Delta t|^2 + 4.278 \times 10^{-5} \, |\Delta t| \cdot |\Delta R_t|$$

and

$$\Delta^2\alpha_t(2) = 3.335 \times 10^{-5} \, |\Delta t|^2 + 4.289 \times 10^{-5} \, |\Delta t| \cdot |\Delta R_t| \, .$$

Thus, their numerical values are, respectively, $\Delta^2\alpha_t(1) = 1.82 \times 10^{-7}$ and $\Delta^2\alpha_t(2) = 1.818 \times 10^{-7}$. Therefore, the numerical values of MRI of second order are $\frac{\Delta^2\alpha_t}{\alpha_t}(1) = 4.866 \times 10^{-5}$ and $\frac{\Delta^2\alpha_t}{\alpha_t}(2) = 4.859 \times 10^{-5}$.

As $\frac{\Delta^2\alpha_t}{\alpha_t}(2) < \frac{\Delta^2\alpha_t}{\alpha_t}(1)$, then the second experiment has smaller inaccuracy with respect to the first one. Therefore, the more accurate value of the temperature coefficient of the two experiments is equal to $\alpha_t(2) \pm \left(\Delta^1\alpha_t(2) + \frac{1}{2}\Delta^2\alpha_t(2)\right) = 0.003742 \pm 2.962 \times 10^{-6} C^{-1}$.

6 Discussion

The suggested by us principle for representation of the maximum inaccuracies of an indirectly measurable variable and method for determining their numerical values are applicable and important to each and every experimental field of science in which the indirectly measurable variable can be represented as a function of directly measurable variables. Moreover:

Our principle is applicable to different fields of science where experiments are conducted, to different mathematical models, to different kinds of measurements (with different utensils and using different methods) and to different types of input data used in the measurement.

In the currently used methods the value of the maximum inaccuracy of an indirectly measurable variable is found as an arithmetical mean of the calculated values of all of its observations of the experiment. Our method is adequate to the objective reality as we calculate the value of the maximum inaccuracy of the indirectly measurable variable as a function of the arithmetic means of its arguments.

Our method can be applied to other experiments modelled by functions which have continuous first derivatives but do not have second derivatives with respect to some of its arguments in given points. Using it one can practically easily calculate the numerical value of the maximum inaccuracy of a given experiment.

In [9], [10], [11] we introduced spaces and hyperplanes of the inaccuracies of first order. Using them we define a scale for determining the quality of an experiment [11]. This method can also be applied to the inaccuracies of second order.

The maximum inaccuracies of second order are more sensitive than the inaccuracies of first order as they are multiplicative and not only additive. This is why they give an opportunity for more precise determination of the inaccuracy when measuring the indirectly measurable variable. This is especially important for the experiments requiring excellent accuracy when determining the results.

7 Conclusion

In the current paper we continue our research on determination of the maximum inaccuracy of an indirectly measurable variable by introducing MAI and MRI of second order. We show how the introduced by us method can be applied to determine the numerical values of the maximum inaccuracy of second order and to functions which have continuous first derivatives but do not have a second derivative with respect to some of its arguments in given points.

The determination of MAI and MRI with quadratic approximation gives more accurate results for the limits in which an indirectly measurable variable is determined.

The natural processes are described by real and continuous functions. However, in the mathematical models, describing these processes, this may not be the case. This is why we should point out that when dealing with such models our principle and method may not be applicable.

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Received February 22, 2015

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