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# SKEW RING EXTENSIONS AND GENERALIZED MONOID RINGS

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**Abstract.** A D-*structure* on a ring $A$ with identity is a family of self-mappings indexed by the elements of a monoid $G$ and subject to a long list of rather natural conditions. The mappings are used to define a generalization of the monoid algebra $A[G]$. We consider two of the simpler types of D-structure. The first is based on a homomorphism from $G$ to $\mathrm{End}(A)$ and leads to a skew monoid ring. We also explore connections between these D-structures and normalizing and subnormalizing extensions. The second type of D-structure considered is built from an endomorphism of $A$. We use D-structures of this type to characterize rings which can be graded by a cyclic group of order 2.

## 1. Introduction

A system called a *D-structure* in [6] and introduced in [5] consists of a ring $A$ with an identity 1, a monoid $G$ with identity $e$ and mappings $\sigma_{x,y}\colon A \to A$ satisfying the following condition for all $x, y, z \in G$ and $a, b \in A$.

*Condition* (A).

(0) For each $x \in G$ and $a \in R$, we have $\sigma_{x,y}(a) = 0$ for almost all $y \in G$.

(i) Each $\sigma_{x,y}$ is an additive endomorphism.

(ii) $\sigma_{x,y}(ab) = \sum_{z\in G} \sigma_{x,z}(a)\sigma_{z,y}(b)$.

(iii) $\sigma_{xy,z} = \sum_{uv=z} \sigma_{x,u} \circ \sigma_{y,v}$.

($\mathrm{iv}_1$) $\sigma_{x,y}(1) = 0$ if $x \neq y$;

($\mathrm{iv}_2$) $\sigma_{x,x}(1) = 1$;

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(iv$_3$) $\sigma_{e,x}(a) = 0$ if $x \neq e$;
(iv$_4$) $\sigma_{e,e}(a) = a$.

For brevity a $D$-structure described in the notation of Condition (A) will be referred to as "a $D$-structure $\sigma$" or by cognate phrases.

In [5] a sort of "skew" or "twisted" monoid ring $A\langle\langle G,\sigma\rangle$ associated with $A$ and $G$ was constructed by means of the mappings $\sigma_{x,y}$. (The connection with the structures usually called skew monoid rings will be elucidated in the next section.) The multiplication in $A\langle G,\sigma\rangle$ is given by the rule

$$(a\cdot x)(b\cdot y) = a\sum_{z\in G}\sigma_{x,z}(b)\cdot zy$$

and distributivity. Examples include group rings, skew polynomial rings and the Weyl algebras. There are also connections with gradings of rings.

We shall examine two relatively simple types of $D$-structures: those defined by a monoid homomorphism from $G$ to the monoid of (ring-) endomorphisms of $A$ and those defined by an endomorphism $f$ of $A$ using the fact that if $\delta(a) = a - f(a)$ for all $a \in A$ then $\delta$ is an $(f,\mathrm{id})$-derivation, i.e. $\delta(ab) = \delta(a)b + f(a)\delta(b)$ for all $a, b \in A$. For the former we establish connections with *normalizing extensions* [3] and *subnormalizing extensions* (also known as *triangular extensions* [7], [11]). We obtain criteria for $\mathbb{Z}_2$-gradability by means of the latter.

## 2. D-structures, skew monoid rings and normalizing extensions

For a ring $R$ with identity 1 and a monoid $G$ with identity $e$ let

$$F\colon G \to \mathrm{End}(A)$$

be a monoid homomorphism, where $\mathrm{End}(A)$ is the monoid of ring endomorphisms with respect to composition. We obtain a $D$-structure $\sigma^F$ by defining $\sigma^F_{x,y}$ to be $F(x)$ if $x = y$ and the zero map otherwise. (This is easily verified; cf. [5, Example 1].) This is the situation where $G$ *acts on* $A$ by endomorphisms.

Proposition 2.1. *If $\sigma$ is defined by a monoid homomorphism $F$, then in $A\langle G,\sigma\rangle$ the multiplication is given by the formula*

$$(a\cdot x)(b\cdot y) = a\sigma(x)(b)\cdot xy.$$

If $\sigma$ is defined by an endomorphism $F$ then Proposition 2.1 says that $A\langle G,\sigma\rangle$ $(= A\langle G,\sigma^F\rangle)$ is the *skew monoid ring of $G$ over $A$* defined by $F$ in the sense of [1]. In particular when $G$ is the free monoid generated by an