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Minimization of Disproportionality in PR Voting Systems

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Abstract: A generalized algorithm for the minimization of disproportionality in voting systems with proportional representation, using one of 10 indices, inclusive Sainte-Laguë, d'Hondt, Lijphart and Gallagher indices, is proposed.
Keywords: algorithm, disproportion, index, optimization, proportional representation, voting systems.

1 Introduction

When taking collective decisions, using voting systems with proportional representation (PR), to minimize the disproportion of deciders' will representation is required – disproportion caused by the character in integers of the number of deciders and that of alternative options. To estimate this disproportion, the use of special indices is needed. In this paper, the use, in this aim, of Sainte-Laguë, d'Hondt, Gallagher, Lijphart, Relative deviation and other five indices in party-list PR (LPR) elections is investigated.

2 The LPR elections' disproportionality minimization problem

Let: M – number of seats in the elected body; n – number of parties that have reached or exceeded the representation threshold; V – total valid votes cast for the n parties; V_i – total valid votes cast for party i; x_i – number of seats to be allocated to party i; I – index of disproportionality. It is required [3] to determine unknowns x_i ($i=\overline{1,n}$) – nonnegative integers, which will assure the I extreme value (minimum or maximum, depending on the essence of I)

$$I = f(M; n; V_i, x_i, i = \overline{1,n}) \rightarrow \text{extremum} \quad (1)$$

in compliance with restrictions:

$$x_1 + x_2 + \ldots + x_n = M, \quad (2)$$

$$V_1 + V_2 + \ldots + V_n = V. \quad (3)$$

D'Hondt index I_H is the minimum ratio between v_i and m_i

$$I_H = \min\{v_i/m_i, i = \overline{1,n}\} = \min\{V_i/(Qx_i), i = \overline{1,n}\}, \quad (4)$$

where $Q = V/M$ (Hare quota [1]), $v_i = 100 \cdot V_i/V$ (%) and $m_i = 100 \cdot x_i/M$ (%).

Sainte-Laguë index [1], noted here $I_{S\text{-}L}$, is calculated as

$$I_{S-L} = \sum_{i=1}^{n} \frac{1}{v_i}(v_i - m_i)^2 \qquad (5)$$

Gallagher index [1], noted here I_{Ga}, is determined as

$$I_{G_a} = \sqrt{\frac{1}{2}\sum_{i=1}^{n}(v_i - m_i)^2} = \sqrt{\frac{1}{2}\sum_{i=1}^{n}\left(\frac{V_i}{V} - \frac{x_i}{M}\right)^2} = \frac{100}{V}\sqrt{\frac{1}{2}\sum_{i=1}^{n}(V_i - Qx_i)^2} \qquad (6)$$

Lijphart index [2], noted here I_L, constituted the maximum absolute deviation between m_i and v_i

$$I_L = \max\{|v_i - m_i|, i = \overline{1,n}\}. \qquad (7)$$

3 The generalized algorithm of disproportionality minimization

Let $a_i = [V_i/Q]$, where $[z]$ signifies the integer part of z. Then $V_i = a_iQ + \Delta V_i$ and $x_i = a_i + \Delta x_i$, where ΔV_i is the remainder from dividing V_i to Q, and Δx_i is the number of supplementary seats for party i, in addition to the a_i ones. It is easy to prove that, in sense of (1)-(7), $\Delta x_i = 0$ or $\Delta x_i = 1$. So, the formula for the minimum value I^*_{Ga} of the **Gallagher index** (6) can be presented as

$$I^*_{G_a} = \frac{100}{V}\min\sqrt{\frac{1}{2}\sum_{i=1}^{n}(Qa_i + \Delta V_i - Qa_i - \Delta x_i Q)^2} =$$

$$\frac{100}{V}\min\sqrt{\frac{1}{2}\sum_{i=1}^{n}(\Delta V_i - \Delta x_i Q)^2}. \qquad (8)$$

Thus, to each party i ($i = \overline{1,n}$) has already been allocated proportionally (with 0 contribution to the value of I^*_{Ga}) by a_i seats, using in this aim a_iQ votes, and remaining undistributed $\Delta M \le n - 1$ seats for ΔMQ unused votes; the problem of minimizing I_{Ga} is reduced to determining $\Delta x_i, i = \overline{1,n}$ in (7). The following equalities hold

$$\Delta M = \Delta x_1 + \Delta x_2 + \cdots + \Delta x_n = (\Delta V_1 + \Delta V_2 + \cdots + \Delta V_n)/Q. \qquad (9)$$

Expression $Q - \Delta V_i = Q - R_j = \Delta R_j \ge 0$ represents the complement of remainder $R_j = \Delta V_i$ to Hare quota Q. Each such complement contributes to the value of I_{Ga} only if $\Delta x_i = 1$, to party i being distributed in excess ΔR_j votes, and does not contribute, if $\Delta x_i = 0$, in this case contributing to the value of I_d the remainder ΔV_i itself, which is equal to the number of votes loosed by party i. Conform to (9), from the n values Δx_i only ΔM are equal to 1. Given that the total number ΔR of votes in excess, allocated to parties for which $\Delta x_i = 1$, is equal to the total number of votes ΔV, loosed by parties for which $\Delta x_i = 0$, we get

$$\Delta R = \sum_{j=1}^{\Delta M}\Delta R_j = \Delta MQ - \sum_{j=1}^{\Delta M} R_j = \sum_{j=\Delta M+1}^{n} R_j = \Delta V. \qquad (10)$$

Replacing (10) in (8), we get

$$I_{G_a}^* = \frac{100}{V}\min\sqrt{\frac{1}{2}\left(\sum_{j=1}^{\Delta M}\Delta R_j^2 + \sum_{j=\Delta M+1}^{n} R_j^2\right)}. \qquad (11)$$

The value of $(Q - \Delta V_i)^2$ is preferable to the $(Q - \Delta V_k)^2$ one, to be one of the ΔM factors of the first sum in (11), only if relation $(Q - \Delta V_i)^2 + \Delta V_k^2 <$ $(Q - \Delta V_k)^2 + \Delta V_i^2$ holds, from where we get the necessary condition: $\Delta V_i <$ ΔV_k. So, to minimize I_{Ga} it is necessary and sufficient that from the n ones to select ΔM largest remainders $R_j = \Delta V_i$ and to each of the respective parties to add by one additional seat to the already allocated a_i ones. It can be seen that the algorithm for minimization of I_{Ga} is similar to the method of largest remainder with the Hare quota [1].

The minimum value I_L^* of **Lijphart index** (7) can be got from formula

$$I_L^* = \min\max_{i=\overline{1,n}}|v_i - m_i| = \frac{100}{V}\min\max_{i=\overline{1,n}}|\Delta V_i - \Delta x_i Q| = \frac{100}{V}\min\Delta R_k,$$

where ΔR_k is the largest from the ΔM smallest complements ΔR_j from the n ones. In the base of suggestions, similar to ones used for the minimization of the Gallagher index, it can be found that the minimization of the Lijphart index is assured by the method of largest remainder with the Hare quota, too.

The seats allocation optimization, in sense of **d'Hondt index** (4), supposes that the smallest from ratios v_i/m_i, $i = \overline{1,n}$ be as large as possible:

$$I_H^* = \max\min\left\{\frac{V_i}{Qx_i}, i = \overline{1,n}\right\} = \max\min\left\{\frac{V_i}{[Q(a_i+\Delta x_i)]}, i = \overline{1,n}\right\}. \qquad (12)$$

By the definition of Q, the following relations hold: $V_i/[Q(a_i + 1)] < 1$ $\leq V_i/(Qa_i)$ and $a_1 + a_2 + \ldots + a_n \leq M < n + a_1 + a_2 + \ldots + a_n$. Equality in the last relation holds only at proportional allocation of seats to the n parties; in this case $I_H^* = 1$. If $I_H^* < 1$, then for $K \leq \Delta M$ parties hold $\Delta x_i^* \geq 1$, and for the other n-K ones – $\Delta x_i^* = 0$. So, when determining $I_H^* < 1$ can be considered only ratios $V_i/[Q(a_i + \Delta x_i)]$, where $\Delta x_i \geq 1$. Moreover, the value of $I_H^* < 1$ is equal to the smallest of the $\Delta M \leq n - 1$ largest ratios $V_i/[Q(a_i + \Delta x_i)]$. Noting U_h - the number of votes cast for the party with such ratio, the expression (12) can be presented as

$$I_H^* = \begin{cases} 1, \text{if } a_1 + a_2 + \cdots a_n = M \\ U_k/[Q(a_k + \Delta x_k)], \text{otherwise} \end{cases}.$$

In a similar mode, it is proved that the minimization of disproportionality: 1) in sense of Sainte-Laguë index (5) and of Standard deviation index

[3], is assured by the same algorithm as used for the d'Hondt index, with the difference that the last ΔM seats are allocated one by one to each of the first K parties with the largest ratios $V_i/[2(a_i + \Delta x_i - 1)+1]$, where $\Delta x_i \geq 1$; 2) in sense of Rae [1], Loosemore-Handby [1], Rose [1], Grofman and Relative deviation [3] indices, is assured by the method of largest remainder with the Hare quota. So, the minimization of disproportionality, in sense of investigated 10 indices, is assured by the following generalized algorithm:

1. $x_i := a_i = [V_i/Q]$, $i = \overline{1, n}$. Determining the number $\Delta M := M - (a_1 + a_2 + \ldots + a_n)$ of still undistributed seats. If $\Delta M = 0$, than the allocation of seats is finished and it is a proportional one.

2. One by one seat, from the ΔM still undistributed ones, is additionally allocated: to the first ΔM parties with the largest remainders ΔV_i, when using the Gallagher, Grofman, Lijphart, Loosemore-Handby, Rae, Rose, or Relative deviation indices; to the first $K \leq \Delta M$ parties with the largest ratios $V_i/[2(a_i + \Delta x_i - 1)+1]$, when using the Sainte-Laguë or Standard deviation indices, or – ratios $V_i/[(a_i+\Delta x_i)]$, when using the d'Hondt index, where $\Delta x_i \geq 1$. Obtained representation is, however, nonproportional, but it guarantees, for the seven of the first group of indices, that to each party it will allocate a number of seats that will exceed or lose the proportion of its votes cast by less than one seat.

The difference of this algorithm from the ones, used by d'Hondt and Sainte-Laguë methods, is that it don't need the consecutive dividing of V_i to divisors $u_i + 1$ ($u_i = 0, 1, 2, \ldots$), as in case of the d'Hondt method, or to divisors $2u_i + 1$ ($u_i = 0, 1, 2, \ldots$), as in case of the Sainte-Laguë method, and the solution is obtained after a considerably smaller number of steps.

4 Conclusion

The proposed generalized algorithm for the minimization of disproportionality, when allocating seats in RPL elections using Sainte-Laguë, d'Hondt, Gallagher, Lijphart or Relative deviation indices, is considerably reducing the needed number of steps if compare to Sainte-Laguë or d'Hondt methods.

References

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