Some Properties of a Lattice Generated by Implicational Logics

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Consider the following implicational formulae:

$$\boldsymbol{A}_1 = (\boldsymbol{p} \supset \boldsymbol{p}), \boldsymbol{A}_2 = (((\boldsymbol{p} \supset \boldsymbol{p}) \supset \boldsymbol{p}) \supset \boldsymbol{p}) = ((\boldsymbol{A}_1 \supset \boldsymbol{p}) \supset \boldsymbol{p}), \cdots,$$
$$\boldsymbol{A}_{i+1} = ((\boldsymbol{A}_i \supset \boldsymbol{p}) \supset \boldsymbol{p}), \cdots, (i = 1, 2, 3, \cdots)$$

Using these formulae (axioms), we may construct the following logics:

$$\boldsymbol{L}_1 =< \boldsymbol{A}_{2i} >, \boldsymbol{L}_2 =< (\boldsymbol{A}_{2i} \supset \boldsymbol{A}_{2i+1}) >,$$
$$\boldsymbol{L}_3 =< \boldsymbol{A}_{2i-1} >, \boldsymbol{L}_4 =< (\boldsymbol{A}_{2i-1} \supset \boldsymbol{A}_{2i}) >, i = 1, 2, 3, \cdots$$

viz. the logic $\boldsymbol{L}_1$ is generated by the axioms $\boldsymbol{A}_{2i}$, $i = 1, 2, 3, ...$; the process is analogous for logics $\boldsymbol{L}_2, \boldsymbol{L}_3, \boldsymbol{L}_4$.
The rule of deduction for these logics is unique - modus ponens: $\boldsymbol{A}, (\boldsymbol{A} \supset \boldsymbol{B}) \vdash \boldsymbol{B}$ (if the formulae $\boldsymbol{A}$ and $(\boldsymbol{A} \supset \boldsymbol{B}) \in$ to the given logic, then formula $\boldsymbol{B}$ also $\in$ to this logic).
Let $\boldsymbol{S}$ be the lattice generated by the logics $\boldsymbol{L}_1, \boldsymbol{L}_2, .\boldsymbol{L}_3, \boldsymbol{L}_4$. The following results are obtained:

1. Lattice $\boldsymbol{S}$ is infinite.

2. If logics $\boldsymbol{L}_1, \boldsymbol{L}_2, .\boldsymbol{L}_3, \boldsymbol{L}_4$ possess a finite number of axioms (viz. i = 1, 2, 3, ..., n), then the respective lattice $\boldsymbol{S}$ is finite.

3. For lattice $\boldsymbol{S}$ the problem of the equality of any two lattice elements is solvable.

4. If the rule of deduction - the substitution is added to the above logics, then statements 1)–3) are also true. (The rule of deduction the substitution means: if formula $\boldsymbol{A} \in$ to the given logic, then the result of the substitution in formula $\boldsymbol{A}$ of any implicational formula of the variable $\boldsymbol{p}$ for the same variable $\boldsymbol{p}$ also $\in$ to the same logic.)