Estimates for Solutions to Partial Quasilinear Differentila Equations

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Let's consider

$$L(u) = a(x,t)u_{tt} + b(x,t)u_{tx} + c(x,t)u_{xx} + d(x,t)u_t + h(x,t)u_x, \quad x \in R^1.$$

The second order quasilinear equations are studied in the following form:

$$L(u) + r(u)\left[a(x,t)u_t^2 + b(x,t)u_t u_x + c(x,t)u_x^2\right] + f(x,t,u) = 0,\ x \in R^1. \tag{1}$$

The objective is to reduce this equation to a linear equation and to study the solutions of the equation (1) depending on the solutions of the linear equation obtained and the functions $r(u)$ and $f(x,t,u)$. For this purpose we make the substitution $u = z(v)$, $v = v(x,y)$. (2)
If $z(v)$ will be a nontrivial solution to the ordinary differential equation $z'' + r(z)(z')^2 = 0$ (3), then equation (1) will be reduced to the following linear equation for determining the function $v(x,y)$: $L(v) + g(x,t)v = 0$, $x \in R^1$ (4), where $f(x,t,z(v)) \cdot (z')^{-1} = g(x,t)v$.
This will specifically be if we take $f \equiv 0$ and the condition $z'(v) \neq 0$ is met.
Here are some particular cases of the substitution $u = z(v)$ of (2) and the function $f(x,t,u)$ of equation (1), for which it is reduced to equation (4)

$$1)\ r(u) = \alpha,\ \alpha \neq 0;\ u = \frac{1}{\alpha}\ln v;\ f = g(x,t);$$

$$2)\ r(u) = -\frac{1}{u};\ u = e^v;\ f = g(x,t)u\ln u;$$

$$3)\ r(u) = \frac{\alpha}{u},\ \alpha \neq -1;\ u = v^{\frac{1}{1+\alpha}};\ f = g(x,t)u^{2\alpha+1};$$

$$4)\ r(u) = \frac{\alpha}{u} + (\alpha+1)u^{\alpha},\ \alpha \neq -1;\ u = \left[\ln v(\alpha+1)\right]^{\frac{1}{1+\alpha}};\ f = g(x,t)u^{\alpha};$$

$$5)\ r(u) = \frac{u}{1-u^2};\ u = \sin v;\ f = g(x,t)\arcsin u \cdot \sqrt{1-u^2}.$$

Thus, if we are able to determine the general (or particular) solution of equation (4), then the general solution (or the corresponding particular one) of equation (1) will be expressed by relation (2). Here are some examples of how to apply this method to the study of the solution to Cauchy's problem for some quasilinear equations.
Example 1. Cauchy's problem for the hyperbolic equation with variable coefficients:

$$L(u) + r(u)\left[u_t^2 - au_x^2\right] + f(u) = 0, \quad L(u) = u_{tt} - au_{xx} + bu_t + cu_x;$$

$$u(x,0) = \varphi_1(x),\ u_t(x,0) = \varphi_2(x). \tag{5}$$

Here the coefficients a, b, c, f are functions that depend on x and t with $a(x,t) \geq a_0 > 0$. By performing the substitution (2), for the function $v(x,t)$ we will get the next problem of Cauchy:

$$L(v) + g(x,t)v = 0, \quad v(x,0) = \psi_1(x), \quad v_t(x,0) = \psi_2(x). \tag{6}$$

I refer to paper [3] where the following theorem is proved: if in the area of $S(x,\tau) = \{x \in R^1,\, 0 \le \tau \le t\}$ function a admits the derivatives bounded up to including order 5 and the functions b, c, g derivatives up to order 4, then for the solution of the problem (6) the following estimation takes place:

$$From\, |\psi_1(x)| \le M_1,\ |\psi_2(x)| \le M_2 \Rightarrow |v(x,t)| \le C_1(t)M_1 + C_2(t)M_2. \tag{7}$$

Thus, by determining the corresponding substitution, we can obtain estimates of the solutions of problem (5) based on the estimates of type (7) for the solution of problem (6). From these estimates results the uniqueness of the solution of problem (5) and the continuous dependence of the solution on the initial conditions of the problem. So, to emphasize: in case 1) the estimates for the solution of the problem (5) are of the type (7), and in other cases these estimates have another type.

This will be made obvious in the case of the equations with constant coefficients, shown in the examples below.

Example 2. Cauchy's problem for the hyperbolic equation:

$$L(u) + r(u)\left[u_t^2 - a^2 u_x^2\right] = 0, \quad L(u) = u_{tt} - a^2 u_{xx};$$

$$u(x,0) = \varphi_1(x),\ u_t(x,0) = \varphi_2(x). \tag{8}$$

Applying the substitution (2) with $f = 0$ for the function $v(x,t)$, we will get Cauchy's problem for the vibrating string equation: $L(v) = 0;\ v(x,0) = \psi_1(x),\ v_t(x,0) = \psi_2(x)$. If the function $\psi_1(x)$ is double derivable and $\psi_2(x)$ derivable, then the solution to this problem is given by the formula of D'Alembert:

$$v(x,t) = \frac{1}{2}\left[\psi_1(x-at) + \psi_1(x+at)\right] + \frac{1}{2a}\int\limits_{x-at}^{x+at} \psi_2(y)dy.$$

$$From\, |\psi_1(x)| \le M_1,\ |\psi_2(x)| \le M_2 \Rightarrow |v(x,t)| \le M_1 + tM_2.$$

Let's examine a couple of particular cases from 1) to 5) examined above and the corresponding estimates that are obtained for the solution to problem (8), emphasizing the suplimentary conditions to the innitial conditions:

$$1)\, r(u) = \alpha;\ u = \frac{1}{\alpha}\ln v;\ \ \psi_1(x) = e^{\alpha\varphi_1(x)};\ \ \psi_2(x) = \alpha\varphi_2(x)e^{\alpha\varphi_1(x)}, a\varphi_2(x) > 0;$$

$$|u(x,t)| \le \left|\frac{1}{\alpha}\ln\left(e^{\alpha M_1} + t|\alpha|M_2 e^{\alpha M_1}\right)\right| \le \left|\frac{1}{\alpha}\left(\alpha M_1 + \ln(1 + t|\alpha|M_2\right)\right| \le M_1 + tM_2.$$

$$2)\, r(u) = -\frac{1}{u};\ u = e^v;\ \psi_1(x) = \ln\varphi_1(x),\ \varphi_1(x) \ge c > 0;\ \psi_2(x) = \frac{\varphi_2(x)}{\varphi_1(x)};$$

$$|u| \le e^{\ln M_1 + \frac{1}{c}tM_2} \le M_1 e^{c_1 t M_2}.$$

$$3)\, r(u) = \frac{n}{u};\ u = \sqrt[n+1]{v};\ \psi_1(x) = \varphi_1^{n+1}(x);\ \psi_2(x) = (n+1)\varphi_1^n(x)\varphi_2(x);$$

$$|u| \le \sqrt[n+1]{M_1 + t(n+1)M_1^n M_2}, with\ \varphi_1(x)\varphi_2(x) \ge 0\ for\ n\ unequal.$$

Example 3. Cauchy's problem for the parabolic equation:

$$L(u) - r(u)a^2u_x^2 = 0, \quad L(u) = u_t - a^2u_{xx};\ u(x,0) = \varphi(x). \tag{9}$$

Applying substitution (2), for function v Cauchy's problem for the heat equation in a homogeneous bar is obtained: $L(v) = 0;\ v(x, 0) = \psi(x)$. If the function $\psi(x)$ is continuous, then the solution to this problem is given by the following formula:

$$v(x,t) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{+\infty} \psi(y) e^{-\frac{(x-y)^2}{4a^2 t}} dy = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \psi\left(x + 2az\sqrt{t}\right) e^{-z^2} dz,$$

because $\int\limits_{-\infty}^{+\infty} e^{-z^2} dz = \sqrt{\pi}$, from $|\psi(x)| \leq M \Rightarrow |v(x,t)| \leq M$.

Note that different specific cases can be examined and the corresponding estimates can be deducted as in the solutions to problem (9).

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